Math Hawker Digest **Secondary 3**

Additional Mathematics – Algebra part I

1. Advanced Algebra **2. Polynomial** **3. Partial Fractions**

1 Advanced Algebra

Let's take algebra to the next level in A-Math.

More Quadratic Factorization

In E-Math we learned to factor quadratics when the leading coefficient is 1. When it isn't, dividing can create messy fractions, so we use a method called "splitting the middle term".

$$ax^2 + bx + c \quad \Rightarrow \quad (ax + m)(x + n)$$

We need to find 2 numbers p and q that satisfies the following:

- $p \times q = a \times c$
- $p + q = b$

Once we do, we can rewrite the middle term to form:

$$ax^2 + px + qx + c$$

We can group the resultant terms to factors the expression into $(ax + m)(x + n)$, where m and n are determined from p and q.

Example 1:

Factorize $3x^2 + x - 10$.

Step 1): Find the appropriate values of p and q.

– $pq = 3(-10)$
$pq = -30$ — (1)

– $p + q = 1$
$p = 1 - q$ — (2)

– Sub (2) into (1):
$(1 - q)q = -30$
$q^2 - q - 30 = 0$
$(q - 6)(q + 5) = 0$
$q = 6 \quad \text{or} \quad q = -5$

– Arbitrarily choose $q = 6$ and sub into (2):
$p = 1 - 6 = -5$

Step 2): Factorize the result.

– $$\begin{aligned} 3x^2 + x - 10 &= 3x^2 - 5x + 6x - 10 \\ &= (3x^2 + 6x) + (-5x - 10) \\ &= 3x(x + 2) - 5(x + 2) \\ &= (3x - 5)(x + 2) \end{aligned}$$

Discriminant

The discriminant is an expression that reveals the nature of roots in a quadratic equation (real or complex, distinct or repeated) without needing to solve the equation.

Discriminant	Nature of Roots
$b^2 - 4ac > 0$	2 distinct real roots
$b^2 - 4ac = 0$	2 equal real root
$b^2 - 4ac < 0$	0 real roots and 2 complex roots

BTW: This might be our first encounter with ***complex*** numbers. Don't worry about them, they are not part of the O-Level syllabus. For now, we only care that no ***real*** roots exist. By ***real***, we mean the regular kinds of numbers we have been using so far.

Example 1:

Find the range of values of k for which the equation $3x^2 - 2x = k - 1$ has 2 distinct real solutions.

– $3x^2 - 2x = k - 1$
$3x^2 - 2x + (1 - k) = 0$

– 2 distinct real roots exist.

⇒ $b^2 - 4ac > 0$
$(-2)^2 - 4(3)(-k + 1) > 0$
$k > \frac{2}{3}$

Example 2:

Find the value of k for which the circle $(x - 2)^2 + (y - 3)^2 = k$ touches the line $y = -x + 5$ at exactly one point.

– $(x - 2)^2 + (y - 3)^2 = k$ — (1)
$y = -x + 5$ — (2)

– Sub (2) into (1):
$(x - 2)^2 + (-x + 5 - 3)^2 = k$
$x^2 - 4x + 4 + x^2 - 4x + 4 = k$
$2x^2 - 8x + (8 - k) = 0$ — (3)

– 1 intersect between (1) and (2).

⇒ (3) has one root.

⇒ $b^2 - 4ac = 0$
$(-8)^2 - 4(2)(8 - k) = 0$
$k = 0$

Example 3:

Find the range of values of k such that $f(x) = (2k - 1)x^2 + 3kx + \frac{1}{4}$ has 2 distinct real roots.

Step 1): Find values of k where $f(x)$ is quadratic.

– $f(x)$ is only quadratic if the leading coefficient is non-zero.

⇒ $2k - 1 \neq 0$
$k \neq \frac{1}{2}$

Step 2): Determine the discriminant D of $f(x)$.

– 2 distinct real roots exist.

⇒ $b^2 - 4ac > 0$
$(3k)^2 - 4(2k - 1)\left(\frac{1}{4}\right) > 0$
$9k^2 - 2k + 1 > 0$

– $D(x) = 9x^2 - 2x + 1$

Step 3): Determine if $D(x) > 0$.

– The leading coefficient of $D(x)$, $9 > 0$, is positive.

⇒ $D(x) \geq 0$

– $b^2 - 4ac = (-2)^2 - 4(9)(1)$
$= -32 < 0$

⇒ $D(x)$ has no real roots.

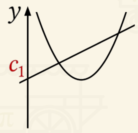
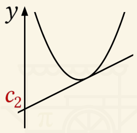
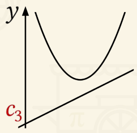
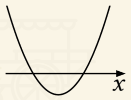
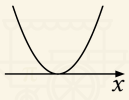
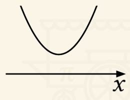
⇒ $D(x)$ does not cross the x-axis.

⇒ $D(x) > 0$

⇒ $f(x)$ has 2 distinct real roots for all values of $k \neq \frac{1}{2}$.

Simultaneous Equations

Solving simultaneous equations involves finding roots, which correspond to points where the graphs of the equations intersect.

Case 1: Two Solutions	Case 2: One Solution	Case 3: No Solutions
$y = ax^2 + bx + c$ $y = mx + c_1$	$y = ax^2 + bx + c$ $y = mx + c_2$	$y = ax^2 + bx + c$ $y = mx + c_3$
		
two intersects	one intersect	no intersects
↓	**Equate and Simplify**	↓
$y = ax^2 + (b-m)x + (c-c_1)$	$y = ax^2 + (b-m)x + (c-c_2)$	$y = ax^2 + (b-m)x + (c-c_3)$
		
two distinct real roots	one distinct real root (two equal real roots)	no real roots (two complex roots)

Example 1:

Solve the simultaneous equations.

$$\begin{cases} y - x = 1, \\ x^2 + y^2 = 25 \end{cases}$$

- $y - x = 1$ — (1)
  $y = x + 1$ — (2)
- $x^2 + y^2 = 25$
- Sub (1) into (2):
  $x^2 + (x+1)^2 = 25$
  $2x^2 + 2x - 24 = 0$
  $2(x+4)(x-3) = 0$
  $x = -4$ or $x = 3$
- Sub $x = -4$ into (1):
  $y = -4 + 1 = -3$
- Sub $x = 3$ into (1):
  $y = 3 + 1 = 4$
- $(x, y) = (-4, -3)$ or $(x, y) = (3, 4)$

Example 2:

a) Given that $f(x) = x^2 - 5x + 6$. Find the range of values such that the line $y = x + k$ does not intersect the curve $y = f(x)$.

b) Determine the value of k and the (x, y) coordinate for which $y = x + k$ is a tangent to the curve $y = f(x)$.

a)
- $y = x^2 - 5x + 6$ — (1)
  $y = x + k$ — (2)
- Sub (1) into (2):
  $x^2 - 5x + 6 = x + k$
  $x^2 - 6x + (6 - k) = 0$ — (3)
- There are no intersects.

⇒ $b^2 - 4ac < 0$
$(-6)^2 - 4(1)(6-k) < 0$
$k < -3$

b)
- $y = x + k$ is tangent to $y = f(x)$.

⇒ There is exactly 1 intersect.

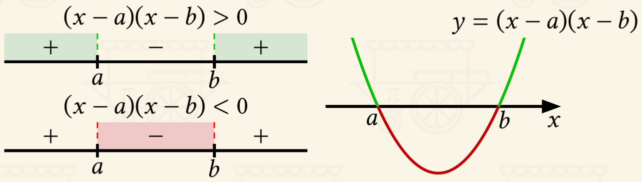
⇒ $b^2 - 4ac = 0$
$(-6)^2 - 4(1)(6-k) = 0$
$k = -3$

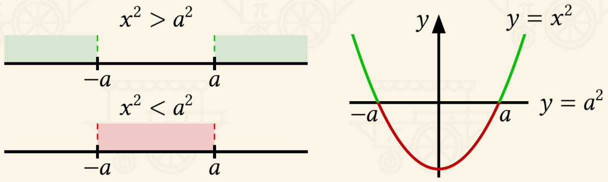
- Sub $k = -3$ into (3):
  $x^2 - 6x + [6 - (-3)] = 0$
  $(x-3)^2 = 0$
  $x = 3$
- Sub $x = 3$, $k = -3$ into (2):
  $y = 3 + (-3) = 0$

∴ Tangency occurs at $k = -3$ with the tangent point at $(3, 0)$.

Quadratic Inequalities

We've used the number line for compound inequalities in E-Math Now, let's apply it to solve quadratic inequalities too.





Example 1:

Determine all real solutions to the following inequalities and represent the solution on a number line.

a) $(x+2)(x-6) \geq 0$ b) $15x - 3x^2 - 12 > 0$

c) $x^2 - 2x - 4 \leq 0$

a) Step 1): Find regions of sign changes.
- Let $F(x) = (x+2)(x-6)$
- Sign changes when $F(x)$ crosses the x-axis.

⇒ $(x+2)(x-6) = 0$
$x = -2$ or $x = 6$

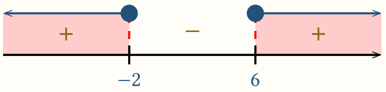
- Boundaries: $x = -2$ and $x = 6$
- Distinct regions: $(\infty, -2], [-2, 6], [6, \infty)$

Step 2): Find the signs of each region by testing values.
- Sub $x = -3$ into $F(x)$:
  $(-3+2)(-3-6) = (-1)(-9) = 9 > 0$
- Sub $x = 0$ into $F(x)$:
  $(0+2)(0-6) = (2)(-6) = -12 < 0$
- Sub $x = 7$ into $F(x)$:
  $(7+2)(7-6) = (9)(1) = 9 > 0$

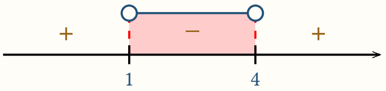
Step 3): Determine regions that satisfy condition.
- $(x+2)(x-6) \geq 0$

⇒ Looking for positive regions (inclusive of 0).

⇒ $x \leq -2$ or $x \geq 6$



b)
- $15x - 3x^2 - 12 > 0$
  $-3(x-1)(x-4) > 0$
  $(x-1)(x-4) < 0$
- Boundaries: $x = 1$ and $x = 4$
- After testing values: $1 < x < 4$



c) – $x^2 - 2x - 4 \le 0$

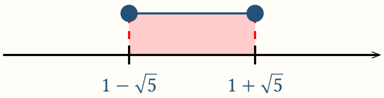
$(x-1)^2 - (-1)^2 - 4 \le 0$

$(x-1)^2 \le 5$

$-\sqrt{5} \le x - 1 \le \sqrt{5}$

$1 - \sqrt{5} \le x \le 1 + \sqrt{5}$

– Boundaries: $x = 1 - \sqrt{5}$ and $x = 1 + \sqrt{5}$



Example 2:

Find the range of values of x for which $4 < (x-3)^2 \le 25$. Represent the solution on a number line.

Step 1): Solve lower bound $(x-3)^2 > 4$.

– $(x-3)^2 > 4$

$x - 3 < -\sqrt{4}$ or $x - 3 > \sqrt{4}$

$x < 1$ or $x > 5$

Step 2): Solve upper bound $(x-3)^2 \le 25$.

– $(x-3)^2 \le 25$

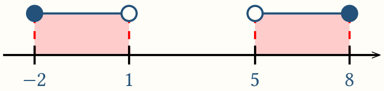
$-\sqrt{25} \le x - 3 \le \sqrt{25}$

$-2 \le x \le 8$

Step 3): Combine the constraints.

i. $x < 1$ or $x > 5$

ii. $-2 \le x \le 8$



– $-2 \le x < 1$ or $5 < x \le 8$

2 Polynomials

Let's go beyond quadratics and explore equations of higher powers with a new set of techniques and theorems.

Polynomials

Polynomials are algebraic expressions comprising terms with non-negative integer powers of x.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

where a_i is the coefficient of the term with exponent i. A reminder that a coefficient is the number in front of a term. For example, the coefficient of the term $5x^3$ is 5.

Polynomial Equality

Two polynomials are equal if and only if their coefficients are equal.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$
$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2 x^2 + b_1 x^1 + b_0$$

$$P(x) = Q(x) \quad \Leftrightarrow \quad a_n = b_n, \ldots, a_0 = b_0$$

Degree of Polynomials

The degree of a polynomial is the highest power of its terms.

$$P(x) = a_n x^{\overset{\text{degree}}{n}} + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$$

The degree of the product of two polynomials is the sum of their degrees.

$$\underset{P(x) \times Q(x)}{\text{degree of}} = \underset{P(x)}{\text{degree of}} + \underset{Q(x)}{\text{degree of}}$$

Example 1:

Given polynomial $P(x) = 2x^3 + 4x^2 + x - 1$, find $P(-3)$.

– $P(-3) = 2(-3)^3 + 4(-3)^2 + (-3) - 1$

$= -54 + 36 - 3 - 1$

$= -22$

Example 2:

Given that $5x^2 + 9x - 2 = (x-1)(ax+4) + (x+1)(bx+2)$ for all real values of x. Find the values of a and b.

Step 1): Evaluate the right-hand side.

$(x-1)(ax+4) + (x+1)(bx+2)$

$= ax^2 + 4x - ax - 4 + bx^2 + 2x + bx + 2$

$= (a+b)x^2 + (b - a + 6)x - 2$

Step 2): Match coefficients of the two sides.

– $a + b = 5$ — (1)

– $b - a + 6 = 9$

$b - a = 3$ — (2)

Step 3): Solve for a and b.

– (1) + (2):

$(a+b) + (b - a + 6) = 5 + 9$

$2b + 6 = 14$

$b = 4$

– Sub $b = 4$ into (1):

$a + 4 = 5$

$a = 1$

Example 3:

Given polynomials $P(x) = x^3 - 2x^2 + x$ and $Q(x) = 2x^2 + 5x - 7$, find $P(x) \times Q(x)$. Hence, verify with this example that the degree of the product of polynomials is the sum of their degrees.

– $P(x) \times Q(x) = (x^3 - 2x^2 + x)(2x^2 + 5x - 7)$

$= 2x^5 + 5x^4 - 7x^3 - 4x^4 - 10x^3 + 14x^2 + 2x^3 + 5x^2 - 7x$

$= 2x^5 + x^4 - 15x^3 + 19x^2 - 7x$

– $\underset{P(x)}{\text{degree of}} = 3$ – $\underset{Q(x)}{\text{degree of}} = 2$ – $\underset{P(x) \times Q(x)}{\text{degree of}} = 5$

– $\underset{P(x) \times Q(x)}{\text{degree of}} = \underset{P(x)}{\text{degree of}} + \underset{Q(x)}{\text{degree of}}$

Polynomial Division

Long division can be used on polynomials to find factors.

dividend = divisor × quotient + remainder

$$\begin{array}{r@{\,}l} & x^2 \quad\ + 1 \\ x + 1 & \overline{)\, x^3 + x^2 + x + 2} \\ & -\, x^3 - x^2 \\ \hline & \qquad\quad\ x + 2 \\ & \qquad\ -\, x - 1 \\ \hline & \qquad\qquad\quad 1 \end{array}$$

Example 1:

Find the quotient and remainder when dividing $2x^3 - x^2 + 5$ by $x + 2$. Express the result as an equation.

$$\begin{array}{r@{\,}l} & 2x^2 \ - 5x + 10 \\ x + 2 & \overline{)\, 2x^3 \ - x^2 \qquad\quad + 5} \\ & -\, 2x^3 - 4x^2 \\ \hline & \quad -\, 5x^2 \\ & \quad\ \ 5x^2 + 10x \\ \hline & \qquad\quad 10x \ + 5 \\ & \qquad -\, 10x - 20 \\ \hline & \qquad\qquad\ -\, 15 \end{array}$$

– $2x^3 - x^2 + 5 = (x+2)(2x^2 - 5x + 10) - 15$

Remainder Theorem

The remainder when a polynomial $P(x)$ is divided by a linear divisor $ax+b$ is given by $P\left(-\frac{b}{a}\right)$.

$$\overset{\text{divident}}{P(x)} = \underset{\text{divisor}}{(ax+b)} \times \overset{\text{quotient}}{Q(x)} + \underset{\text{remainder}}{P\left(-\frac{b}{a}\right)}$$

Factor Theorem

The value $x = -\frac{b}{a}$ is a root of a polynomial $P(x)$ if and only if

$$P\left(-\frac{b}{a}\right) = 0$$

This means:

- If $P\left(-\frac{b}{a}\right) = 0$, then $(ax+b)$ is a factor of $P(x)$.
- If $(ax+b)$ is a factor of $P(x)$, then $x = -\frac{b}{a}$ is a solution.

Example 1:

Consider polynomial $P(x) = 3x^3 - 2x^2 + 5x - 7$. Determine the remainder when $P(x)$ is divided by $2x-3$.

$$\begin{aligned} - \quad \text{Remainder} &= P\left(\frac{3}{2}\right) \\ &= 3\left(\frac{3}{2}\right)^3 - 2\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) - 7 \\ &= \frac{81}{8} - \frac{9}{2} + \frac{15}{2} - 7 \\ &= \frac{49}{8} \end{aligned}$$

Example 2:

Given polynomial $P(x) = x^2 + (k-1)x + (k^2-k-2)$, find the value of k for which $P(x)$ is exactly divisible by $x-2$ but not divisible by $x+1$.

$$\begin{aligned} - \quad & P(x) \text{ is divisible by } x-2. \\ \Rightarrow \quad & P(2) = 0 \\ & 2^2 + (k-1)(2) + (k^2-k-2) = 0 \\ & k^2 + k = 0 \\ & k(k+1) = 0 \\ & k = 0 \quad \text{or} \quad k = -1 \\ - \quad & P(x) \text{ is not divisible by } x+1. \\ \Rightarrow \quad & P(-1) \neq 0 \\ & (-1)^2 - (k-1) + (k^2-k-2) \neq 0 \\ & k^2 - 2k \neq 0 \\ & k(k-2) \neq 0 \\ & k \neq 0 \quad \text{and} \quad k \neq 2 \\ - \quad & \text{Combine the conditions:} \\ & \text{i.} \quad k = 0 \quad \text{or} \quad k = -1 \\ & \text{ii.} \quad -k \neq 0 \quad \text{and} \quad k \neq 2 \\ - \quad & k = -1 \end{aligned}$$

Cubic Equations

A cubic expression is a degree-3 polynomial that always has 3 roots. The roots can be one of these combinations:

1) 3 distinct real roots
2) 2 equal real roots and 1 distinct real root
3) 3 equal real roots
4) 1 real root and 2 complex roots

$$\begin{gathered} ax^3 + bx^2 + cx + d = 0 \\ a(x+r_1)(x+r_2)(x+r_3) = 0 \\ x = -r_1 \quad \text{or} \quad x = -r_2 \quad \text{or} \quad x = -r_3 \end{gathered}$$

Cubic Identities

Cubic identities provide useful shortcuts to expand or factorize cubic expressions, simplifying complex algebraic manipulations.

Cube of Binomial

1) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
2) $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Sum of Cubes

1) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

Difference of Cubes

1) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Example 1:

Using the Sum of Cubes principle show that the Difference of Cubes principle $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ is true.

$$\begin{aligned} - \quad x^3 - y^3 &= x^3 + (-y)^3 \\ &= [x + (-y)][x^2 - x(-y) + (-y)^2] \\ &= (x-y)(x^2 + xy + y^2) \end{aligned}$$

Example 2:

Factorize the following polynomials.

a) $64x^3 - 125y^3$ b) $27 - 8(x-1)^3$

$$\begin{aligned} \text{a)} \quad - \quad & 64x^3 - 125y^3 \\ &= (4x)^3 + (-5y)^3 \\ &= (4x-5y)[(4x)^2 - (4x)(-5y) + (-5y)^2] \\ &= (4x-5y)(16x^2 + 20xy + 25y^2) \end{aligned}$$

$$\begin{aligned} \text{b)} \quad - \quad & 27 - 8(x-1)^3 \\ &= 3^3 + [-2(x-1)]^3 \\ &= [3 - 2(x-1)][3^2 + 6(x-1) + 4(x-1)^2] \\ &= (5-2x)(9 + 6x - 6 + 4x^2 - 8x + 4) \\ &= (5-2x)(4x^2 - 2x + 7) \end{aligned}$$

Factorizing Cubic Equations

Factorizing quadratics is relatively straightforward with practice, but not so much for cubics. We need a systematic trial-and-error approach, guided by the **rational root theorem**.

Rational Root Theorem

Given a polynomial with integer coefficients:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

If $P(x)$ has a rational root, then it has the form $\frac{p}{q}$, where:

- p is a factor of the constant term a_0.
- q is a factor of the leading coefficient a_n.

TLDR: To find a factor of a cubic equation $f(x)$, test all factors p_i of the constant term a_0 to see if $f(p_i) = 0$.

Example 1:

a) Factorize completely $f(x) = 3x^3 - 5x^2 - 12x + 20$.

b) Hence, solve $g(y) = 3y^6 - 5y^4 - 12y^2 + 20$

a) Step 1): Find a factor using rational root theorem.

$$\begin{aligned} - \quad & \text{Some possible rational roots to test:} \\ & \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20 \\ - \quad & \text{Test } x = 1 \text{ as a root:} \\ & f(1) = 3(1)^3 - 5(1)^2 - 12(1) + 20 = 6 \neq 0 \\ - \quad & \text{Test } x = -1 \text{ as a root:} \\ & f(-1) = 3(-1)^3 - 5(-1)^2 - 12(-1) + 20 = 24 \neq 0 \\ - \quad & \text{Test } x = 2 \text{ as a root:} \\ & f(2) = 3(2)^3 - 5(2)^2 - 12(2) + 20 = 0 \\ \Rightarrow \quad & x = 2 \text{ is a root of } f(x). \\ \Rightarrow \quad & (x-2) \text{ is a factor of } f(x). \\ \Rightarrow \quad & f(x) = (x-2)(ax^2 + bx + c) \end{aligned}$$

Step 2): Find a, b, and c via long division.

$$\begin{array}{rl} & 3x^2 \quad + x - 10 \\ x-2) & \overline{3x^3 - 5x^2 - 12x + 20} \\ & \underline{-\ 3x^3 + 6x^2} \\ & x^2 - 12x \\ & \underline{-\ x^2 \ + 2x} \\ & -\ 10x + 20 \\ & \underline{\ \ 10x - 20} \\ & 0 \end{array}$$

- $f(x) = (x-2)(3x^2 + x - 10)$

Step 3): Factorize quadratic expression.

- $3x^2 + x - 10$
  $= 3x^2 + 6x - 5x - 10$
  $= (3x^2 + 6x) + (-5x - 10)$
  $= 3x(x+2) - 5(x+2)$
  $= (3x-5)(x+2)$
- $f(x) = (x-2)(x+2)(3x-5)$

b)
- $f(x) = (x-2)(x+2)(3x-5)$
  $x = 2$ or $x = -2$ or $x = \dfrac{5}{3}$
- Let $z = y^2$:
  $g(y) = 3y^6 - 5y^4 - 12y^2 + 20$
  $= 3z^3 - 5z^2 - 12z + 20$
- $z = 2$
  $y^2 = 2$
  $y = \pm\sqrt{2}$
- $z = -2$
  $y^2 = -2$
  (rejected, $y^2 < 0$)
- $z = 5/3$
  $y^2 = 5/3$
  $y = \pm\sqrt{5/3}$
- $y = \pm\sqrt{2}$ or $y = \pm\sqrt{\dfrac{5}{3}}$

3 Partial Fractions

Partial fractions let us break down complicated fractions into simpler ones, making calculations much easier.

Introduction to Partial Fractions

Partial fractions are the simpler fractions obtained by breaking down a complex fraction, making them easier to work with.

We start by revisiting the addition of algebraic fractions:

$$\begin{aligned} & \frac{2}{x+2} + \frac{1}{x-1} \\ &= \frac{2(x-1)}{(x+2)(x-1)} + \frac{1(x+2)}{(x+2)(x-1)} \\ &= \frac{2(x-1) + 1(x+2)}{(x+2)(x-1)} \\ &= \frac{3x}{x^2 + x - 2} \end{aligned}$$

decomposition

Partial fraction decomposition reverses this process by starting with a combined fraction and splitting it into simpler parts.

$$\frac{3x}{x^2 + x - 2} \quad \Rightarrow \quad \frac{2}{x+2} + \frac{1}{x-1}$$

The specific technique used for decomposition depends on the nature of the initial denominator.

Case 1: Distinct Linear Factors

Denominator contains distinct linear factors.

$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

Case 2: Repeated Linear Factors

Denominator contains repeated linear factors.

$$\frac{P(x)}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{P(x)}{(x-a)^3} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$$

Example 1: Express $\dfrac{2x+6}{(x-1)(x-2)}$ in partial fractions.

- $\dfrac{2x+6}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$ — (1)
  $2x + 6 = A(x-2) + B(x-1)$ — (2)
- Sub $x = 1$ into (2):
  $2(1) + 6 = A(1-2) + B(1-1)$
  $A = -8$
- Sub $x = 2$ into (2):
  $2(2) + 6 = A(2-2) + B(2-1)$
  $B = 10$
- Sub $A = -8,\ B = 10$ into (1):
  $\dfrac{2x+6}{(x-1)(x-2)} = -\dfrac{8}{x-1} + \dfrac{10}{x-2}$

Example 2: Express $\dfrac{6}{(x-1)^2(x-2)}$ in partial fractions.

- $\dfrac{6}{(x-1)^2(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$ — (1)
  $6 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$ — (2)
- Sub $x = 1$ into (2):
  $6 = A(1-1)(1-2) + B(1-2) + C(1-1)^2$
  $B = -6$
- Sub $x = 2$ into (2):
  $6 = A(2-1)(2-2) + B(2-2) + C(2-1)^2$
  $C = 6$
- Sub $x = 3$ into (2):
  $6 = A(3-1)(3-2) + B(3-2) + C(3-1)^2$
  $6 = 2A + B + 4C$ — (3)
- Sub $B = -6,\ C = 6$ into (3):
  $6 = 2A + (-6) + 4(6)$
  $A = -6$
- Sub $A = -6,\ B = -6,\ C = 6$ into (1):
  $\dfrac{6}{(x-1)^2(x-2)} = \dfrac{6}{x-2} - \dfrac{6}{x-1} - \dfrac{6}{(x-1)^2}$

Case 3: Irreducible Quadratic Factors

Denominator contains quadratic factors that cannot be factored further into linear terms.

$$\frac{P(x)}{x^2 + bx + c} = \frac{Ax + B}{x^2 + bx + c}$$

Case 4: Repeated Irreducible Quadratic Factors

Denominator contains repeated irreducible quadratic factors.

$$\frac{P(x)}{(x^2 + bx + c)^2} = \frac{Ax + B}{x^2 + bx + c} + \frac{Cx + D}{(x^2 + bx + c)^2}$$

Example 1:

Express $\dfrac{x^3+x+1}{(x^2+1)(x^2+x+1)}$ is partial fractions.

- $$\frac{x^3+x+1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1} \quad \text{— (1)}$$
  $$\begin{aligned} x^3+x+1 &= (Ax+B)(x^2+x+1)+(Cx+D)(x^2+1) \\ &= Ax^3+Ax^2+Ax+Bx^2+Bx+B \\ &\quad +Cx^3+Cx+Dx^2+D \\ &= (A+C)x^3+(A+B+D)x^2 \\ &\quad +(A+B+C)x+(B+D) \end{aligned}$$
- Equate coefficients:
  $$\begin{aligned} 1 &= A+C && \text{— (2)} \\ 0 &= A+B+D && \text{— (3)} \\ 1 &= A+B+C && \text{— (4)} \\ 1 &= B+D && \text{— (5)} \end{aligned}$$
- (3) − (5):
  $$\begin{aligned} 0-1 &= (A+B+D)-(B+D) \\ A &= -1 \end{aligned}$$
- Sub $A=-1$ into (2):
  $$\begin{aligned} 1 &= -1+C \\ C &= 2 \end{aligned}$$
- Sub $A=-1,\ C=2$ into (4):
  $$\begin{aligned} 1 &= -1+B+2 \\ B &= 0 \end{aligned}$$
- Sub $B=0$ into (5):
  $$\begin{aligned} 1 &= 0+D \\ D &= 1 \end{aligned}$$
- Sub $A=-1,\ B=0,\ C=2,\ D=1$ into (1):
  $$\begin{aligned} \frac{x^3+x+1}{(x^2+1)(x^2+x+1)} &= \frac{-1x+0}{x^2+1} + \frac{2x+1}{x^2+x+1} \\ &= \frac{2x+1}{x^2+x+1} - \frac{x}{x^2+1} \end{aligned}$$

Case 5: Improper Fractions

All of the previous cases only apply to proper algebraic fractions. Polynomial long division is used for decomposition if the numerator degree equals or exceeds the denominator degree.

$$\underset{\text{divisor}}{\overset{\text{dividend}}{\frac{P(x)}{Q(x)}}} = \underset{\text{quotient}}{H(x)} + \underset{\text{divisor}}{\overset{\text{remainder}}{\frac{R(x)}{Q(x)}}}$$

Example 1:

Express $\dfrac{4x^3-2x+6}{(x+1)(x-1)}$ as proper partial fractions.

Step 1): Perform long division.

- $$\frac{4x^3-2x+6}{(x+1)(x-1)} = \frac{4x^3-2x+6}{x^2-1}$$
  $$\begin{array}{r} 4x \\ x^2-1\overline{)\;\;4x^3-2x+6} \\ \underline{-\,4x^3+4x} \\ 2x+6 \end{array}$$
- $$\frac{4x^3-2x+6}{(x+1)(x-1)} = 4x + \frac{2x+6}{(x+1)(x-1)}$$

Step 2): Perform partial fractions.

- $$\frac{2x+6}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$
  $$2x+6 = A(x-1)+B(x+1)$$
- Sub $x=1$ into (2):
  $$\begin{aligned} 2(1)+6 &= A(1-1)+B(1+1) \\ B &= 4 \end{aligned}$$
- Sub $x=-1$ into (2):
  $$\begin{aligned} 2(-1)+6 &= A(-1-1)+B(-1+1) \\ A &= -2 \end{aligned}$$
- Sub $A=-2,\ B=4$ into (1):
  $$\frac{2x+6}{(x+1)(x-1)} = \frac{-2}{x+1} + \frac{4}{x-1}$$

Step 3): Combine the results.

- $$\frac{4x^3-2x+6}{(x+1)(x-1)} = 4x + \frac{4}{x-1} - \frac{2}{x+1}$$

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