# Additional Mathematics – Algebra part II

**1. Binomial Theorem** **2. Surds** **3. Exponents & Logarithms**

## 1 Binomial Theorem

A shortcut for a common mathematical form.

### Binomial Theorem

A binomial is a polynomial with two terms. The Binomial Theorem gives a formula to expand powers of binomials without multiplying them out manually.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

where $\binom{n}{r}$ (read "n choose r") stands for "number of ways to choose $r$ elements from $n$ elements" with the following formula:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

### Pascal's Triangle

Pascal's Triangle provides a simple way to visualize the coefficients from the Binomial Theorem.

| | |
|---|---|
| $n=0$ | 1 |
| $n=1$ | 1 1 |
| $n=2$ | 1 2 1 |
| $n=3$ | 1 3 3 1 |
| $n=4$ | 1 4 6 4 1 |
| $n=5$ | 1 5 10 10 5 1 |
| $n=6$ | 1 6 15 20 15 6 1 |

Recall the quadratic and cubic identities. Let's also include the degree-4 expansion which is now simple to compute using the Binomial Theorem.

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

**Example 1:**
Evaluate the expression $\binom{13}{5}$.

– $$\binom{13}{5} = \frac{13!}{5!(13-5)!} = \frac{6227020800}{120 \times 40320} = 1287$$

**Example 2:**
Show that the equation $\binom{n}{n-r} = \binom{n}{r}$ is true.

– $$\binom{n}{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

**Example 3:**
Expand the expression $\left(2x + \frac{3}{2y}\right)^5$.

– $$\left(2x+\frac{3}{2y}\right)^5 = \binom{5}{0}(2x)^5\left(\frac{3}{2y}\right)^0 + \binom{5}{1}(2x)^{5-1}\left(\frac{3}{2y}\right)^1 + \binom{5}{2}(2x)^{5-2}\left(\frac{3}{2y}\right)^2 + \binom{5}{3}(2x)^{5-3}\left(\frac{3}{2y}\right)^3 + \binom{5}{4}(2x)^{5-4}\left(\frac{3}{2y}\right)^4 + \binom{5}{5}(2x)^{5-5}\left(\frac{3}{2y}\right)^5$$
$$= 32x^5 + 120x^4y^{-1} + 180x^3y^{-2} + 135x^2y^{-3} + \frac{405}{8}xy^{-4} + \frac{243}{32}y^{-5}$$

**Example 4:**
Find the coefficient of the term containing $x^2$ in the expansion of $\left(3x^2 - \frac{1}{x}\right)^{10}$.

– $$\binom{10}{6}(3x^2)^{10-6}\left(-\frac{1}{x}\right)^6 = 210 \times 81 \times x^8 \times x^{-6} = 17010x^2$$

– Coefficient is 17, 010.

**Example 5:**
The first two non-zero terms in the expansion of $(1+bx)(1+ax)^4$ in ascending powers of $x$ are 1 and $-5x^2$. Find the value of each of the constants $a$ and $b$, where $a < b$.

Step 1): Expand expression.

– $$(1+ax)^4 = 1 + \binom{4}{1}ax + \binom{4}{2}a^2x^2 + \ldots = 1 + 4ax + 6a^2x^2 + \ldots$$

– $$(1+bx)(1+4ax+6a^2x^2) = 1 + 4ax + 6a^2x^2 + bx + 4abx^2 + \ldots = 1 + (4a+b)x + (6a^2+4ab)x^2 + \ldots$$

Step 2): Match coefficients to information provided.

– $4a + b = 0$, $b = -4a$ — (1)   – $6a^2 + 4ab = -5$ — (2)

– Sub (1) into (2):
$$6a^2 + 4a(-4a) = -5$$
$$a = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

– Sub $a = \pm\frac{\sqrt{2}}{2}$ into (1):
$$b = -4\left(\pm\frac{\sqrt{2}}{2}\right) = \mp 2\sqrt{2}$$
$$(a,b) = \left(\frac{\sqrt{2}}{2}, -2\sqrt{2}\right) \quad \text{or} \quad (a,b) = \left(-\frac{\sqrt{2}}{2}, 2\sqrt{2}\right)$$

– Since $a < b$,
$$(a,b) = \left(-\frac{\sqrt{2}}{2}, 2\sqrt{2}\right)$$

# 2 Surds

Simplifying expressions with square roots is not trivial. Surds help us with this without dealing with pesky decimals.

## Laws of Surds

The laws of surds help us manipulate and combine rooted expressions efficiently.

**Laws**

1) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
2) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

**Special Cases**

1) $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$
2) $\sqrt{a^2 b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$

**Negative Cases**

1) $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
2) $\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$

Remember these laws only apply when $a, b > 0$, as square roots of negative numbers are not defined in the real number system.

*BTW:* The term "surds" comes from the Latin word ***surdus***, meaning "deaf" or "mute". Rational numbers were thought of as "spoken" numbers because they could be expressed as exact ratios, while irrational numbers (like $\pi$) were considered "mute" or "inexpressible" in those terms.

**Example 1:**

Simplify the following expressions, leaving your solution in surd form.

a) $\sqrt{12} \times \sqrt{63}$ b) $\dfrac{\sqrt{72}\sqrt{20}}{\sqrt{2}\sqrt{8}}$

c) $\sqrt{27} + \sqrt{75}$

a) – $\sqrt{12} \times \sqrt{63} = \sqrt{2^2 \times 3} \times \sqrt{3^2 \times 7}$
$= 2\sqrt{3} \times 3\sqrt{7}$
$= 6\sqrt{21}$

b) – $\dfrac{\sqrt{72}\sqrt{20}}{\sqrt{2}\sqrt{8}} = \dfrac{\sqrt{3^2 \times 2^2 \times 2}\sqrt{2^2 \times 5}}{\sqrt{16}}$
$= \dfrac{6\sqrt{2} \times 2\sqrt{5}}{4}$
$= 3\sqrt{10}$

c) – $\sqrt{27} + \sqrt{75} = 3\sqrt{3} + 5\sqrt{3}$
$= 8\sqrt{3}$

**Example 2:**

Expand $(2\sqrt{5} - 4)(3 - \sqrt{40})$, leaving your solution in surd form.

– $(2\sqrt{5} - 4)(3 - \sqrt{40}) = (2\sqrt{5} - 4)(3 - 2\sqrt{10})$
$= 6\sqrt{5} - 4\sqrt{50} - 12 + 8\sqrt{10}$
$= 6\sqrt{5} - 20\sqrt{2} - 12 + 8\sqrt{10}$

**Example 3:**

A rectangle has an area of is $(24 + b\sqrt{2})\ \text{m}^2$. Given that it's length is $(a + 3\sqrt{2})$ m and breadth is $(6 - 2\sqrt{2})$ m, find the values $a$ and $b$.

– Area = Length × Breadth
$= (a + 3\sqrt{2})(6 - 2\sqrt{2})$
$= 6a - 2a\sqrt{2} + 18\sqrt{2} - 6(2)$
$= (6a - 12) + (18 - 2a)\sqrt{2}$

– $6a - 12 = 24$
$a = 6$

– $18 - 2(6) = b$
$b = 6$

## Conjugates

Multiplying conjugate surds eliminates the square root using the difference of squares.

$$\underbrace{(p + q\sqrt{a})(p - q\sqrt{a})}_{\text{conjugates}} = p^2 - q^2 a$$

*BTW:* The term "conjugate" in mathematics generally means a paired counterpart to an object, where the pair has a special relationship or symmetry under certain operations. The word comes from the Latin word ***conjugare***, meaning "to join together". In different contexts, conjugates serve various roles but share the idea of being complementary or related by some transformation.

**Example 1:**

Expand $(2\sqrt{5} - 7)(2\sqrt{5} + 7)$, leaving your solution in surd form.

– $(2\sqrt{5} - 7)(2\sqrt{5} + 7) = 2^2 \times 5 - 7^2 = -29$

## Rationalization

Having square roots in denominators can quickly make algebraic manipulation messy. Conjugates help us get rid of them.

$$\frac{1}{p + q\sqrt{a}} = \frac{1}{p + q\sqrt{a}} \times \underbrace{\frac{p - q\sqrt{a}}{p - q\sqrt{a}}}_{=1} = \frac{p - q\sqrt{a}}{p^2 - q^2 a}$$

**Example 1:**

Rationalize and simplify the following expressions.

a) $\dfrac{2}{\sqrt{2}}$ b) $\dfrac{2}{\sqrt{3} - 1}$

a) – $\dfrac{2}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

b) – $\dfrac{2}{\sqrt{3} - 1} = \dfrac{2}{\sqrt{3} - 1} \times \dfrac{\sqrt{3} + 1}{\sqrt{3} + 1} = \dfrac{2(\sqrt{3} + 1)}{3 - 1} = \sqrt{3} + 1$

**Example 2:**

Solve $\dfrac{x}{\sqrt{5}} = x - \sqrt{20}$, leaving your solution in a rational surd form.

– $\dfrac{x}{\sqrt{5}} = x - \sqrt{20}$
$x = \sqrt{5}x - \sqrt{100}$
$x(1 - \sqrt{5}) = -10$
$x = \dfrac{10}{\sqrt{5} - 1} = \dfrac{10(\sqrt{5} + 1)}{5 - 1} = \dfrac{5(\sqrt{5} + 1)}{2}$

# 3 Exponents & Logarithms

To the moon!

## Laws of Indices (Recap)

Here's a quick recap of the laws of indices covered in E-Math.

**Definition**

1) $a^m = \underbrace{a \times a \times \ldots \times a \times a}_{m}$

**Special Exponents**

1) $a^0 = 1$
2) $a^{-n} = \dfrac{1}{a^n}$
3) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

**Laws**

1) $a^m \times a^n = a^{m+n}$
2) $\dfrac{a^m}{a^n} = a^{m-n}$
3) $(a^m)^n = a^{m \times n}$
4) $a^n \times b^n = (a \times b)^n$
5) $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

## Euler's Number

Euler's number, denoted as $e$, is one of the most important mathematical constants. Like how $\pi$ is closely tied to circles, $e$ is connected to exponential functions. Also like $\pi$, $e$ is an irrational number.

$$e = 2.718281828459045$$

We'll often see $e$ pop up when we deal with exponents and logarithms. There is further discussion at the end of the chapter.

**Example 1:**

Solve the following equations.

a) $9^{2x} = 27^{14-x}$ b) $2^x - 2^{x-3} = \frac{7}{4}$

c) $\left(\frac{1}{25}\right)^{1-x} + (\sqrt{5})^{4x} = 130$ d) $e^{1-x^2} = 1$

a) – $9^{2x} = 27^{14-x}$
$3^{2(2x)} = 3^{3(14-x)}$
$2(2x) = 3(14-x)$
$x = 6$

b) – $2^x - 2^{x-3} = \frac{7}{4}$
$2^x - \frac{1}{2^3}2^x = \frac{7}{4}$
$\left(1 - \frac{1}{8}\right)2^x = \frac{7}{4}$
$\frac{7}{8}2^x = \frac{7}{4}$
$2^x = 2^1$
$x = 1$

c) – $\left(\frac{1}{25}\right)^{1-x} + (\sqrt{5})^{4x} = 130$
$\frac{1}{5^2}5^{2x} + 5^{\frac{1}{2}4x} = 26 \times 5$
$\left(\frac{1}{5^2} + 1\right)5^{2x} = 26 \times 5$
$\left(\frac{26}{5^2}\right)5^{2x} = 26 \times 5$
$5^{2x} = 5^3$
$x = \frac{3}{2}$

d) – $e^{1-x^2} = 1$
$e^{1-x^2} = e^0$
$1 - x^2 = 0$
$x = \pm 1$

## Logarithms

A logarithm is essentially the opposite of an exponent. It is the number of multiples (the index) of a number (the base) to get another number.

**Definitions**

1) $y = b^x \Leftrightarrow x = \log_b y$ (y: value, b: base, x: index)
2) $\lg a = \log_{10} a$
3) $\ln a = \log_e a$

**Laws**

1) $\log_b x + \log_b y = \log_b xy$
2) $\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$
3) $\log_b (x^r) = r \log_b x$

**Properties**

1) $\log_b 1 = 0$
2) $\log_b b = 1$
3) $\log_b b^x = x$
4) $b^{\log_b a} = a$

**Uniqueness of Powers**

1) $b^p = b^q \Leftrightarrow p = q$
2) $\log_b p = \log_b q \Leftrightarrow p = q$

**Change of Base**

1) $\log_b a = \frac{\log_c a}{\log_c b}$
2) $\log_b a = \frac{1}{\log_a b}$

*BTW:* "Logarithm" comes from the Greek words "ratio" (***logos***) and "number" (***arithmos***). John Napier introduced logarithms in the 16th century based on ratios of geometric sequences. Two hundred years later, Euler formalized them, linking them to the exponential function and the constant $e$.

**Example 1:**

Solve the following equations.

a) $45 = 10^x$ b) $\ln(4x) = 5$

c) $\log_x(x+2) = 2$

a) – $45 = 10^x$
$\lg(45) = \lg(10^x)$
$\lg(45) = x\lg(10)$
$\lg(45) = x \times 1$
$x = \lg(45) = 1.65$

b) – $\ln(4x) = 5$
$4x = e^5$
$x = \frac{e^5}{4} = 37.1$

c) – $\log_x(x+2) = 2$
$x + 2 = x^2$
$x^2 - x - 2 = 0$
$(x-2)(x+1) = 0$
$x = 2$ or $x = -1$

**Example 2:**

Solve the following equations.

a) $9^{2x} = 27^{14-x}$ b) $e^{1-x^2} = 1$

a) – $9^{2x} = 27^{14-x}$
$3^{2(2x)} = 3^{3(14-x)}$
$\log_3(3^{4x}) = \log_3(3^{42-3x})$
$(4x)\log_3(3) = (42-3x)\log_3(3)$
$(4x)1 = (42-3x)1$
$x = 6$

b) – $e^{1-x^2} = 1$
$\ln(e^{1-x^2}) = \ln(1)$
$(1-x^2)\ln(e) = 0$
$1 - x^2 = 0$
$x = \pm 1$

**Example 3:**

Solve $\log_3 \sqrt{e^x} = \log_{81}(e^x + 6)$.

– $\log_3(\sqrt{e^x}) = \log_{81}(e^x + 6)$
$\log_3(e^{\frac{x}{2}}) = \frac{\log_3(e^x+6)}{\log_3(81)}$
$= \frac{\log_3(e^x+6)}{\log_3(3^4)}$
$= \frac{\log_3(e^x+6)}{4}$
$4\log_3(e^{\frac{x}{2}}) = \log_3(e^x+6)$
$\log_3(e^{2x}) = \log_3(e^x+6)$
$e^{2x} = e^x + 6$

– Let $y = e^x$.
$y^2 = y + 6$
$y^2 - y - 6 = 0$
$(y-3)(y+2) = 0$
$y = 3$ or $y = -2$

– $e^x = -2$ (rejected, $e^x < 0$)

– $e^x = 3$
$x = \ln(3)$

**Example 4:**

The population of a certain bacteria colony decreases over time due to a lack of nutrients. The population $P$, at time $t$ hours is given by:

$$P = 5000e^{kt}$$

where $k$ is a constant.

a) What was the initial population of the bacteria?

b) Given the population halves after 4 hours, find the value of $k$.

c) Determine the time when the population is one-tenth of its original size.

a) – $P = 5000e^{kt}$ — (1)
– Sub $t = 0$ into (1):
$P_0 = 5000e^{k(0)}$
$= 5000$ bacteria

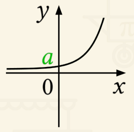
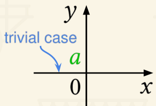
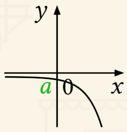
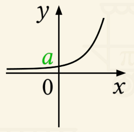
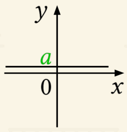
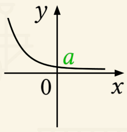
b) – Sub $t = 4$, $P = \frac{5000}{2}$ into (1):
$\frac{5000}{2} = 5000e^{k(4)}$
$\frac{1}{2} = e^{4k}$
$\ln\left(\frac{1}{2}\right) = 4k$
$k = \frac{-\ln(2)}{4} = -0.173$

c) – Sub $P = \frac{5000}{10}$ into (1):
$\frac{5000}{10} = 5000e^{-\frac{\ln(2)}{4}t}$
$\ln\left(\frac{1}{10}\right) = -\frac{\ln(2)}{4}t$
$t = \frac{4\ln(10)}{\ln(2)} = 13.3$ hours

## Exponential Curves

Let's expand on the introduction to exponential curves in E-Math.

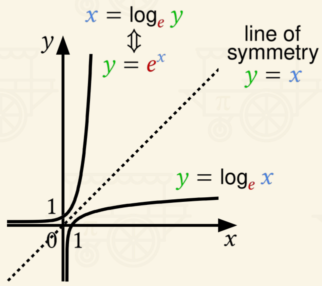
$$y = ab^x$$

| $a > 0$ | $a = 0$ or $b = 0$ | $a < 0$ |
|---|---|---|
|  |  |  |
| $y > 0$ | $y = 0$ | $y < 0$ |
| $b > 1$ | $b = 1$ | $0 < b < 1$ |
|  |  |  |
| gradient $> 0$ | gradient $= 0$ | gradient $< 0$ |

$y = ab^x$ is undefined on the real-number domain for $b < 0$.

## Curve Symmetry

The symmetry of exponential and logarithmic functions is visually apparent from their curves.




The exponential curve starts from (but doesn't touch) the $x$-axis and extends to infinity, while the logarithmic curve starts from (but, again, doesn't touch) the $y$-axis and extends to infinity. They are reflections of each other across $y = x$.

## More on Euler's Number

Bernoulli first discovered the constant $e$ while investigating compound interest. Later, Euler uncovered its deeper connections to exponentials, many of which we will encounter throughout our O-Level and A-Level syllabi.

**Logarithms**

- $\ln(e) = 1$
- Just covered.

**Differentiation**

- $\frac{d}{dx}e^x = e^x$
- Covered later in A-Math.

**Complex Numbers**

- $e^{i\pi} + 1 = 0$
- Covered in A-Levels H2 Math.

**Compound Interest**

- $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$
- Touched upon in E-Math, covered in A-Levels H2 Math.

**Infinite Series**

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- Covered in University Calculus.

There is no hidden meaning behind $e$ beyond the fact that it is the unique value for which these mathematical properties hold. In the same way that $\pi$ is simply the ratio that relates a circle's radius to its area and circumference, $e$ naturally emerges in problems involving exponential growth, calculus, and limits.

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