Math Hawker Digest **Secondary 3**

Additional Mathematics – Lines

1. Coordinate Geometry **2. Lines** **3. Circles**

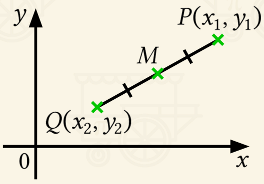
1 Coordinate Geometry

More useful techniques for dealing with coordinates and points!

Midpoint of Line Segment

This is how we find the point coordinates exactly halfway between two given points.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Example 1:

Find the coordinates of the midpoint of the line segment that joins $P(3, 1)$ and $Q(-2, 5)$.

$$\text{Midpoint}_{PQ} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{3 + (-2)}{2}, \frac{1 + 5}{2}\right) = \left(\frac{1}{2}, 3\right)$$

Example 2:

Given $M(6, 3)$ is the midpoint of the line segment that joins $P(a, -2)$ and $Q(9, b)$, find a and b.

- $\left(\frac{a + 9}{2}, \frac{-2 + b}{2}\right) = (6, 3)$
- $\frac{a + 9}{2} = 6$
  $a = 3$
- $\frac{-2 + b}{2} = 3$
  $b = 8$

Example 3:

$A(4, -5)$, $B(9, 3)$, and $C(0, 6)$ are vertices of a rhombus $ABCD$.

a) Find the midpoint M of AC.

b) Find the coordinates of D.

c) Find the area of rhombus $ABCD$.

a) – $M = \left(\frac{4 + 0}{2}, \frac{-5 + 6}{2}\right) = \left(2, \frac{1}{2}\right)$

b) – Let $D = (x_D, y_D)$
$\left(\frac{9 + x_D}{2}, \frac{3 + y_D}{2}\right) = \left(2, \frac{1}{2}\right)$

– $\frac{9 + x_D}{2} = 2$
$x_D = -5$

– $\frac{3 + y_D}{2} = \frac{1}{2}$
$y_D = -2$

– $D = (-5, -2)$

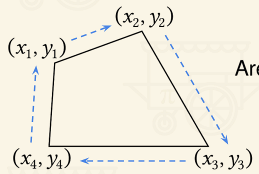
c) – $AC = \sqrt{(4 - 0)^2 + (-5 - 6)^2}$
$= 11.7$ units

– $BD = \sqrt{[9 - (-5)]^2 + [3 - (-2)]^2} = 14.8$ units

– $\text{Area}_{ABCD} = \frac{1}{2} \times 11.7 \times 14.8 = 87.0 \text{ units}^2$

Area of Polygons

The Shoelace Method finds the area of any simple polygon (a polygon with no self-intersections and no holes).



listed in clockwise order

$$\text{Area} = \frac{1}{2} \times \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1)]$$

The calculated area will be negative if the points are listed in anticlockwise order. To obtain a positive and meaningful result, ensure the points are arranged in clockwise order.

Example 1:

Find the area of the triangle with vertices $A(-6, 3)$, $B(2, 4)$, and $C(9, 1)$.

$$\text{Area}_{ABC} = \frac{1}{2} \times \begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} -6 & 2 & 9 & -6 \\ 3 & 4 & 1 & 3 \end{vmatrix}$$

$$= \frac{1}{2}\,|[(-6)(4) + (2)(1) + (9)(3)] - [(3)(2) + (4)(9) + (1)(-6)]|$$

$$= \frac{1}{2}\,|-24 + 2 + 27 - 6 - 36 + 6|$$

$$= \frac{1}{2}\,|-31|$$

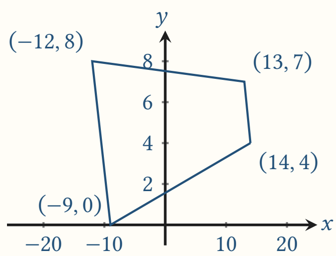
$$= \frac{31}{2}$$

$$= 15.5 \text{ units}^2$$

Example 2:

Find the area of the quadrilateral with vertices $A(14, 4)$, $B(13, 7)$, $C(-12, 8)$, and $D(-9, 0)$.

Step 1): Plot quadrilateral to determine order of vertices.



Step 2): Apply the Shoelace Method.

$$\text{Area}_{ABCD} = \frac{1}{2}\begin{vmatrix} 14 & 13 & -12 & -9 & 14 \\ 4 & 7 & 8 & 0 & 4 \end{vmatrix}$$

$$= \frac{1}{2}\,|[(14)(7) + (13)(8) + (-12)(0) + (-9)(4)] - [(4)(13) + (7)(-12) + (8)(-9) + (0)(14)]|$$

$$= \frac{1}{2}\,|98 + 104 + 0 - 36 - 52 + 84 + 72 - 0|$$

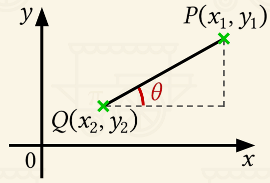
$$= 135 \text{ units}^2$$

More on Gradients

Let's take a closer look at gradients while tying in some concepts from geometry.

Gradient Formula

- $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{Opp}}{\text{Adj}} = \tan\theta$

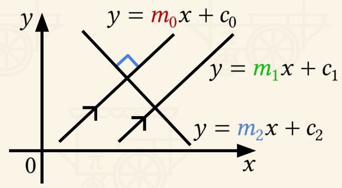


Parallel Lines

- $m_1 = m_0$

Perpendicular Lines

- $m_2 = -\dfrac{1}{m_0}$



Collinearity

Points are collinear if they lie on the same line. Here are some ways to test for it.

Trivial Case

1) $n = 2$ points are always collinear.

Horizontal and Vertical Lines

1) Points that share the same x-coordinate are collinear and form a vertical line.
2) Points that share the same y-coordinate are collinear and form a horizontal line.

Slope Check (general test)

1) Select a reference point.
2) Calculate the gradient between every other point and the reference point.
3) Points that share the same gradient relative to the reference point form a collinear subset (together with the reference).

Area of Polygon (alternative test)

1) Plot points and list points in clockwise order (if $n > 3$).
2) Calculate the area of polygon using the Shoelace Method.
3) If the area is 0, points are collinear.

Example 1:

The coordinates of 5 points are $A(-1,-1)$, $B(2,2)$, $C(2,5)$, $D(4,9)$, and $E(6,0)$.

a) Identify 2 sets of collinear points.

b) Find the angle between the lines formed by the 2 sets of points.

a) – Choose point $A(-1,-1)$ as the reference point.

– $m_{AB} = \dfrac{-1-2}{-1-2} = 1$ – $m_{AD} = \dfrac{-1-9}{-1-4} = 2$

– $m_{AC} = \dfrac{-1-5}{-1-2} = 2$ – $m_{AE} = \dfrac{-1-0}{-1-6} = \dfrac{1}{7}$

– $m_{AC} = m_{AD}$

⇒ Points A, C, and D are collinear. Points B and E are collinear.

b) – $m_{BE} = \dfrac{2-0}{2-6} = -\dfrac{1}{2} = -\dfrac{1}{m_{ACD}}$

⇒ $BE \perp ACD$

⇒ The angle between the lines is 90°.

Example 2:

Given 4 points $A(1,2)$, $B(k+3,k+4)$, $C(2k+1,k+6)$, and $D(3k+5,k+8)$, find the value of k such that

a) AB is parallel to CD.

b) A, B, and C are collinear.

a) – $AB \parallel CD$

⇒ $m_{AB} = m_{CD}$

$$\frac{2-(k+4)}{1-(k+3)} = \frac{(k+6)-(k+8)}{(2k+1)-(3k+5)}$$
$$\frac{-k-2}{-k-2} = \frac{-2}{-k-4}$$
$$k = -2$$

b) – Choose point $A(1,2)$ as the reference point.

– Points A, B, and C are collinear.

⇒ $m_{AB} = m_{BC}$

$$\frac{2-(k+4)}{1-(k+3)} = \frac{(k+4)-(k+6)}{(k+3)-(2k+1)}$$
$$\frac{-k-2}{-k-2} = \frac{-2}{-k+2}$$
$$k = 4$$

2 Lines

Lines are one of the simplest geometric objects, yet they lead to surprisingly deep mathematical results.

Point Slope Form

We've seen how quadratic equations take different forms for different uses. Similarly, we'll explore a more general form of a line equation that focuses on its properties at a point rather than just the y-intercept.

Slope-intercept Form

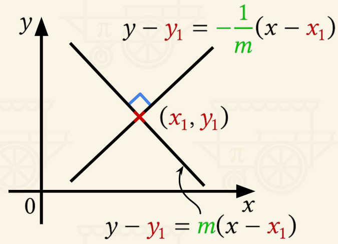
- $y = mx + c$

Point-slope Form

- $y - y_1 = m(x - x_1)$

Perpendicular Line at a Point

- $y - y_1 = -\dfrac{1}{m}(x - x_1)$



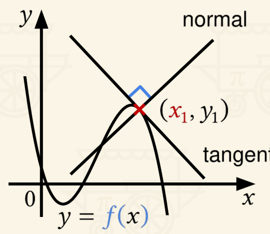
This form also makes working with tangents and their perpendicular counterparts (known as normals) more convenient when dealing with curves.

Tangent at a Point

- $y - f(x_1) = m(x - x_1)$

Normal at a Point

- $y - f(x_1) = -\dfrac{1}{m}(x - x_1)$



Example 1:

Find the equation of a line that passes through the points $A(-2,-9)$ and $B(-4,1)$.

– $m = \dfrac{-9-1}{-2-(-4)} = -5$

– $y - y_A = m(x - x_A)$

$y - (-9) = -5[x - (-2)]$

$y = -5x - 19$

Example 2:

Find the equation of the perpendicular bisector of the line segment joining $A(0,7)$ and $B(3,2.5)$.

– $m_{AB} = \dfrac{7-2.5}{0-3} = -\dfrac{3}{2}$

– $m_{\text{bisector}} = -\dfrac{1}{m_{AB}} = \dfrac{2}{3}$

– $\text{Midpoint} = \left(\dfrac{0+3}{2}, \dfrac{7+2.5}{2}\right) = \left(\dfrac{3}{2}, \dfrac{19}{4}\right)$

– $y - \dfrac{19}{4} = \dfrac{2}{3}\left(x - \dfrac{3}{2}\right)$

$y = \dfrac{2}{3}x + \dfrac{15}{4}$

Linear Models

Lines form the foundation of many advanced models today. By structuring equations in a linear form, we can leverage linear techniques to analyze complex non-linear relationships.

$$Y = MX + C$$

Variable Terms	Constant Terms
y^2 $\log x$ $\frac{1}{\sqrt{x}}$	10 0.812 $\frac{1}{12}$
$y + \frac{a}{x}$ yx	$\frac{m}{\sqrt{c}}$ $a^2 b$

Example 1:

For each of the following equations, where a, b, c, d, and k are constants and x and y are variables, transform them into the linear form $Y = AX + B$.

a) $ax + by = x^2 y$ b) $y = x^3 + ax^2 + b$

c) $y = \frac{x^3}{a + bx}$ d) $y = \frac{ax^b}{c + de^{-kx}}$

a) – $ax + by = x^2 y$
$a + b\frac{y}{x} = xy$
$xy = b\frac{y}{x} + a$

b) – $y = x^3 + ax^2 + b$
$y - x^3 = ax^2 + b$

c) – $y = \frac{x^3}{a + bx}$
$a + bx = \frac{x^3}{y}$
$\frac{x^3}{y} = bx + a$

d) – $y = \frac{ax^b}{c + de^{-kx}}$
$\frac{1}{y} = \frac{c + de^{-kx}}{ax^b}$
$\frac{x^b}{y} = \frac{c}{a} + \frac{d}{a}e^{-kx}$
$\ln\left(\frac{x^b}{y} - \frac{c}{a}\right) = \ln\left(\frac{d}{a}e^{-kx}\right)$
$\ln\left(\frac{x^b}{y} - \frac{c}{a}\right) = \ln\left(\frac{d}{a}\right) - kx$
$\ln\left(\frac{x^b}{y} - \frac{c}{a}\right) = -kx + \ln\left(\frac{d}{a}\right)$

Example 2:

Scientist Rebecca is studying the cooling of a cup of coffee. According to Newton's Law of Cooling, the temperature T°C of the coffee at time t minutes follows the equation

$$T = T_r + (T_0 - T_r)e^{-kt}$$

where T_0 is the initial temperature of the coffee, T_r is the room temperature, and k is a cooling constant.

Rebecca records that the coffee starts at 90°C in a room at 25°C. After 10 minutes, the temperature is 60°C.

a) Represent the equation in a linear form $Y = AX + B$ given the initial conditions.

b) Determine the cooling constant k.

c) Predict the temperature of the coffee after 20 minutes.

a) – $T = T_r + (T_0 - T_r)e^{-kt}$ — (1)
– Sub $T_0 = 90$ and $T_r = 25$ into (1):
$T = 25 + (90 - 25)e^{-kt}$
$T - 25 = 65e^{-kt}$
$\ln(T - 25) = \ln(65e^{-kt})$
$\ln(T - 25) = \ln(65) + \ln(e^{-kt})$
$\ln(T - 25) = \ln(65) - kt$
$\ln(T - 25) = -kt + \ln(65)$ — (2)

b) – Sub $t = 10$ and $T = 60$ into (2):
$\ln(60 - 25) = -k(10) + \ln(65)$
$k = \frac{\ln(65) - \ln(35)}{10} = 0.0619$

c) – Sub $t = 20$ and $k = 0.0619$ into (2):
$\ln(T - 25) = -(0.0619)(20) + \ln(65)$
$T = 25 + e^{\ln(65) - 0.124} = 43.8°C$

Example 3:

The velocity v m/s of a liquid flowing through a pipe is related to the radius r of the pipe by the equation

$$v = kr^n$$

where k and n are constants that depend on fluid and pipe type.

a) Represent the equation in a linear form $Y = AX + B$.

Engineer Iris conducts experiments and obtains the following measurements.

r	0.5	1.0	2.0
v	1.8	4.3	9.7
$\ln(r)$			
$\ln(v)$			

b) Fill in the missing values in the table and plot a line using the data.

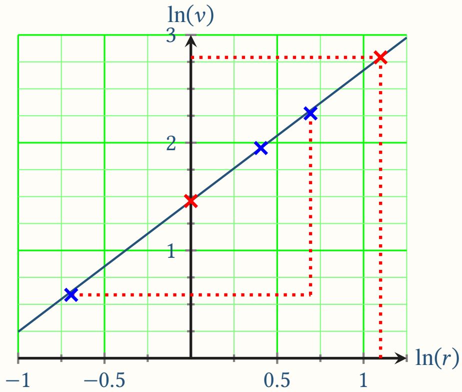
c) Determine the values of k and n.

d) Predict the velocity when the pipe radius is 3.0 m.

a) – $v = kr^n$
$\ln(v) = \ln(kr^n)$
$\ln(v) = \ln(k) + \ln(r^n)$
$\ln(v) = n\ln(r) + \ln(k)$

b)

r	0.5	1.5	2.0
v	1.8	4.3	9.7
$\ln(r)$	−0.693	0.405	0.693
$\ln(v)$	0.588	1.95	2.27



c) – $k = e^{y\text{-intersect}} \approx e^{1.46} = 4.3$
– $n = \text{gradient} = \frac{2.27 - 0.588}{0.693 - (-0.693)} = 1.21$

d) – $\ln(3.0) = 1.10$
– $v_{3.0} \approx e^{2.79} = 16.3$

3 Circles

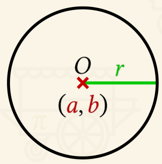
Hell yeah, circles!

Equation of Circle

Here's how we can mathematically represent circles.

$$(x - a)^2 + (y - b)^2 = r^2$$

Center: (a, b) Radius: r



Example 1:

Find the center coordinates and radius of the circle $2x^2 + y^2 + 8x - 3y - 5 = 0$.

- $2x^2 + y^2 + 8x - 3y - 5 = 0$
  $[2x^2 + 8x] + [y^2 - 3y] = 5$
  $2[(x+2)^2 - 2^2] + \left[\left(y - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] = 5$
  $2(x+2)^2 - 8 + \left(y - \frac{3}{2}\right)^2 - \frac{9}{4} = 5$
  $2(x+2)^2 + \left(y - \frac{3}{2}\right)^2 = 5 + 8 + \frac{9}{4}$
  $2(x+2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{\sqrt{61}}{2}\right)^2$
- Center: $\left(-2, \frac{3}{2}\right)$
- Radius: $\frac{\sqrt{61}}{2}$

Example 2:

The points $A(6, -7)$ and $B(8, 4)$ lie on the circumference of a circle. Given that AB forms the diameter, find the radius and center coordinates of the circle.

- Midpoint $= \left(\frac{6+8}{2}, \frac{-7+4}{2}\right) == \left(7, -\frac{3}{2}\right)$
- Diameter $= \sqrt{(6-8)^2 + (-7-4)^2} = \sqrt{125}$
- Center: $\left(7, -\frac{3}{2}\right)$
- Radius: $\frac{\sqrt{125}}{2}$

Example 3:

A circle that passes through the points $A(8, 6)$ and $B(4, 10)$. The circle center lies on the line $y = -2x + 17$. Find the equation of the circle.

- $(x-a)^2 + (y-b)^2 = r^2$ — (1)
- $y = -2x + 16$ — (2)
- Sub $(x, y) = (8, 6)$ into (1):
  $(8-a)^2 + (6-b)^2 = r^2$
  $a^2 - 16a + b^2 - 12b + 100 = r^2$ — (3)
- Sub $(x, y) = (4, 10)$ into (1):
  $(4-a)^2 + (10-b)^2 = r^2$
  $a^2 - 8a + b^2 - 20b + 116 = r^2$ — (4)
- (3) − (4):
  $-16a + 8a - 12b + 20b + 100 - 116 = 0$
  $-8a + 8b - 16 = 0$
  $b = a + 2$
- Sub $x = a$ and $y = b = a + 2$ into (2):
  $a + 2 = -2a + 16$
  $a = 5$
  $b = 5 + 2 = 7$
- Sub $(a, b) = (5, 7)$ and $(x, y) = (8, 6)$ into (1):
  $(8-5)^2 + (6-7)^2 = r^2$
  $10 = r^2$
  $r = \sqrt{10}$ (radius can't be negative)
- The equation of circle is $(x-5)^2 + (y-7)^2 = 10$.

Example 4:

The equation of a circle is given by $(x-2)^2 + (y+3)^2 = 25$. A point $P(5, -7)$ lies on the circle.

a) Find the equation of the tangent to the circle at P

b) Determine whether the line $y = \frac{4}{3}x - \frac{17}{3}$ is a tangent to the circle.

a)
- $m_{\text{radius}} = \frac{-7-(-3)}{5-2} = -\frac{4}{3}$
- $m_{\text{tangent}} = -\frac{1}{m_{\text{radius}}} = \frac{3}{4}$
- $y - 1 = \frac{3}{4}(x - 5)$
  $y = \frac{3}{4}x - \frac{11}{4}$

b)
- $y = \frac{4}{3}x - \frac{17}{3}$ — (1)
- $(x-2)^2 + (y+3)^2 = 25$ — (2)
- Sub (1) into (2):
  $(x-2)^2 + \left[\left(\frac{4}{3}x - \frac{17}{3}\right) + 3\right]^2 = 25$
  $(x-2)^2 + \left(\frac{4}{3}x - \frac{8}{3}\right)^2 = 25$
  $(x-2)^2 + \frac{4^2}{3^2}(x-2)^2 = 25$
  $(x-2)^2\left(1 + \frac{16}{9}\right) = 25$
  $(x-2)^2 = 9$
  $x = -1$ or $x = 5$

⇒ Line intersects the circle at 2 points.

⇒ Line is not a tangent.

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