Math Hawker Digest — Secondary 3

Elementary Mathematics – Algebra

1. Quadratic Equations **2. Inequalities** **3. Indices**

1 Quadratic Equations

Quadratic (Latin for "square") equations model many things, from a ball's flight to a rocket's launch. How do we solve them?

Dual Solution

Let's recall the basics. Dealing with squares often involves two solutions: a positive and a negative one.

Solving $x^2 = c$,

$$x = \pm\sqrt{c}$$
$$x = \sqrt{c} \quad \text{or} \quad x = -\sqrt{c}$$

There is a refresher on mathematical terminology at the end of these notes to ensure we're on the same page linguistically.

Example 1:

Solve the equation $(x-7)^2 = 4$.

$$\begin{aligned} &(x-7)^2 = 4 \\ &x - 7 = \pm\sqrt{4} \\ &x = 7 \pm 2 \\ &x = 9 \quad \text{or} \quad x = 5 \end{aligned}$$

Factorization

Some quadratic equations can be neatly factorized, allowing us to find the solutions straightaway.

Solving $x^2 + bx + c = 0$,

$$x^2 + \underbrace{(m+n)}_{b}x + \underbrace{(m \times n)}_{c} = 0$$
$$(x+m)(x+n) = 0$$
$$x = -m \quad \text{or} \quad x = -n$$

Example 1:

Solve the following equations.

a) $x(x+2) = 3x + 2$ b) $\dfrac{6}{x} - 1 = 2x + 3$

a)
$$\begin{aligned} &x(x+2) = 3x+2 \\ &x^2 - x - 2 = 0 \\ &(x-2)(x+1) = 0 \\ &x = 2 \quad \text{or} \quad x = -1 \end{aligned}$$

b)
$$\begin{aligned} &\frac{6}{x} - 1 = 2x + 3 \\ &6 - x = 2x^2 + 3x \\ &6 - 4x - 2x^2 = 0 \\ &x^2 + 2x - 3 = 0 \\ &(x+3)(x-1) = 0 \\ &x = -3 \quad \text{or} \quad x = 1 \end{aligned}$$

Completing the Square

This method rewrites a quadratic expression as a perfect square, often used when the expression doesn't factor easily.

quadratic expansion: $(x+b)^2 = x^2 + 2bx + b^2$

completing the square: $x^2 + 2bx = (x+b)^2 - b^2$

rearrange

Example 1:

Solve the equation $2x^2 - 6x - 4 = 0$.

$$\begin{aligned} &2x^2 - 6x - 4 = 0 \\ &x^2 - 3x = 2 \\ &\left[x + \left(-\frac{3}{2}\right)\right]^2 - \left(-\frac{3}{2}\right)^2 = 2 \\ &\left[x - \frac{3}{2}\right]^2 = 2 + \frac{9}{4} \\ &x - \frac{3}{2} = \pm\sqrt{\frac{17}{4}} \\ &x = \frac{3}{2} \pm \sqrt{\frac{17}{4}} \\ &x = 3.56 \quad \text{or} \quad x = -0.56 \end{aligned}$$

Quadratic Formula

Derived from the "completing the square" method, the quadratic formula gives a direct method for solving any quadratic equation.

Solving $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1:

Solve the equation $3x^2 - 11x + 5 = 0$.

$$\begin{aligned} &3x^2 - 11x + 5 = 0 \\ &x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(3)(5)}}{2(3)} \\ & = \frac{11 \pm \sqrt{61}}{6} \\ &x = 3.14 \quad \text{or} \quad x = 0.53 \end{aligned}$$

Example 2:

Show that the quadratic formula can be derived using the "completing the square" method.

$$\begin{aligned} &ax^2 + bx + c = 0 \\ &x^2 + \frac{b}{a}x = -\frac{c}{a} \\ &\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} \\ &\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\ &x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \\ &x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Example 3:

Solve the equation $\dfrac{5}{x-3} - \dfrac{x+4}{4} = 3x$.

Step 1): Ensure the domain is well-defined by excluding values that make a denominator 0.

$$\frac{5}{x-3} \quad \Rightarrow \quad x \neq 3$$

Step 2): Solve the equation.

$$\begin{aligned} &\frac{5}{x-3} - \frac{x+4}{4} = 3x \\ &5(4) - (x+4)(x-3) = 3x(4)(x-3) \\ &13x^2 - 35x - 32 = 0 \end{aligned}$$

$$x = \frac{-(-35) \pm \sqrt{(-35)^2 - 4(13)(-32)}}{2(13)}$$
$$= \frac{35 \pm \sqrt{2889}}{26}$$
$$x = 3.41 \quad \text{or} \quad x = -0.72 \quad (\text{both } x \neq 3.)$$

Example 4:

Whitney plans a 75 km journey from Amsterdam to Rotterdam. She can use a bus or a car to travel between the cities. The car's average speed is 20 km/h faster than the bus's. Let the bus's average speed be x km/h.

a) In terms of x, write down how long (in hours) it would take Whitney to reach Rotterdam if they go by car.

b) If they traveled by bus instead of car, they would arrive exactly 1 hour later. Formulate an equation to represent this fact, and show that it simplifies to $x^2 + 20x - 1500 = 0$.

c) By solving the equation, find Whitney's travel time by car in hours and minutes.

a)
- Let s_{car} = Speed of car
  t_{car} = Time taken by car
  d = Distance of journey
- $s_{car} = x + 20$ km/h
- $t_{car} = \frac{d}{s_{car}} = \frac{75}{x+20}$ h

b)
- $t_{bus} = \frac{75}{x}$ h
- $t_{bus} - t_{car} = 1$
  $\frac{75}{x} - \frac{75}{x+20} = 1$
  $75(x+20) - 75(x) = x(x+20)$
  $x^2 + 20x - 1500 = 0$

c)
- $x^2 + 20x - 1500 = 0$
  $(x+50)(x-30) = 0$
  $x = 30$ (bus speed can't be negative)
- $t_{car} = \frac{75}{x+20}$
  $= \frac{75}{30+20}$
  $= 1.5$ h
  $= 1$ hour 30 minutes

2 Inequalities

Hey, where'd the equal signs go?

Solving Inequalities

Inequalities are solved similarly to equations, but they produce a range of solutions instead of a single value. Remember to flip the inequality sign whenever you multiply or divide by a negative number!

Solving $ax + b < c$,

If $a > 0$, then $x < \frac{c-b}{a}$ If $a < 0$, then $x > \frac{c-b}{a}$

Example 1:

Determine the values of x that satisfy the following equations.

a) $4x - 6 < 3x$ b) $3x + 11 \geq 5x - 5$

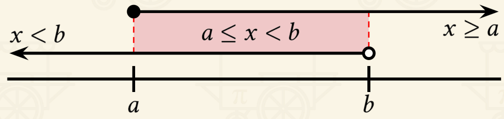
a)
- $4x - 6 < 3x$
  $4x - 3x < 6$
  $x < 6$

b)
- $3x + 11 \geq 5x - 5$
  $3x - 5x \geq -5 - 11$
  $-2x \geq -16$
  $x \leq 8$

Compound Inequalities

Compound inequalities involve multiple conditions identified with overlaps on a number line.

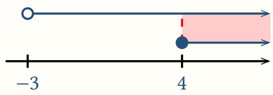
1) Arrows indicate the ranges satisfying each condition.
2) Shaded regions show where both conditions are satisfied.
3) Open circles mean the endpoint is not included; filled circles mean it is.



Example 1:

Determine the values of x for which $4x + 1 > 3x - 2 \geq x + 6$.

- $4x + 1 > 3x - 2 \geq x + 6$
- $4x + 1 > 3x - 2$ and - $3x - 2 \geq x + 6$
  $x > -3$ $x \geq 4$

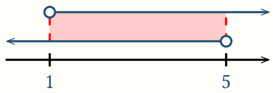


- $x \geq 4$

Example 2:

Determine the values of x for which $3x + 9 < 7x + 5 < 5x + 15$.

- $3x + 9 < 7x + 5 < 5x + 15$
- $3x + 9 < 7x + 5$ and - $7x + 5 < 5x + 15$
  $x > 1$ $x < 5$

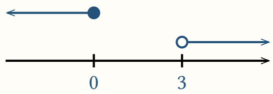


- $1 < x < 5$

Example 3:

Determine the values of x for which $2x + 5 \leq x + 5 < 4x - 4$.

- $2x + 5 \leq x + 5 < 4x - 4$
- $2x + 5 \leq x + 5$ and - $x + 5 < 4x - 4$
  $x \leq 0$ $x > 3$



- No solution.

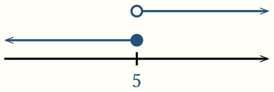
Example 4:

Wen Yi plans to donate 10 hampers to a local charity. She can choose between a large hamper that costs \$25 each or a small hamper that costs \$15 each. She does not want the total cost of all hampers to exceed \$200. In addition, Wen Yi would like to donate more large hampers than small hampers. Can she meet these conditions?

Step 1): Set up constraints.
- Let l = Number of large hampers
  s = Number of small hampers
- $l + s = 10$
  $s = 10 - l$ — (1)
- $25l + 15s \leq 200$ — (2)
- $l > s$ — (3)

Step 2): Solve simultaneous inequalities.
- Sub (1) into (2):
  $25l + 15(10 - l) \leq 200$
  $10l \leq 50$
  $l \leq 5$
- Sub (1) into (3):
  $l > 10 - l$
  $l > 5$

Step 3): Combine the constraints.



– There are no solutions that satisfy both $l \leq 5$ and $l > 5$.

∴ She cannot meet these conditions.

3 Indices

Indices blow numbers up exponentially.

Laws of Indices

The laws of indices simplify operations that involve powers.

Definition

1) $a^m = \underbrace{a \times a \times ... \times a \times a}_{m}$

Special Exponents

1) $a^0 = 1$

2) $a^{-n} = \dfrac{1}{a^n}$

3) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Laws

1) $a^m \times a^n = a^{m+n}$

2) $\dfrac{a^m}{a^n} = a^{m-n}$

3) $(a^m)^n = a^{m\times n}$

4) $a^n \times b^n = (a \times b)^n$

5) $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

Example 1:

Simplify the following expressions.

a) $\left(5^3 \times \dfrac{1}{5^2}\right)^3$ b) $\dfrac{4^{5/2} \times 2^2}{(-3)^4}$ c) $\dfrac{(x^3y)^2 \times \frac{y^2}{z^4}}{x^2z}$

a) – $\left(5^3 \times \dfrac{1}{5^2}\right)^3$

$= (5^3 \times 5^{-2})^3$

$= 5^{(3-2)\times 3}$

$= 5^3$

$= 125$

b) – $\dfrac{4^{5/2} \times 2^2}{(-3)^4}$

$= \dfrac{(2^2)^{5/2} \times 2^2}{(-1)^4 \times (3)^4}$

$= \dfrac{2^{5+2}}{3^4}$

$= 1.58$

c) – $\dfrac{(x^3y)^2 \times \frac{y^2}{z^4}}{x^2z}$

$= (x^6y^2)(y^2z^{-4})(x^2z)^{-1}$

$= x^{6-2}y^{2+2}z^{-4-1}$

$= x^4y^4z^{-5}$

Compound Interest

Compound interest is calculated using powers to represent exponential growth in investments or loans.

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

- P : initial amount (principal)
- r%: interest rate
- n : number of time points
- A_n: total amount after n time points

This equation applies only to compound interest, where interest is earned on both the principal and accumulated interest. Simple or other forms of interest use different formulas.

Example 1:

Rachael places S$2, 000 in the bank for 4 years at 5% per annum, compounded annually. How much does she have at the end of the 4 years?

– $A_4 = 2000\left(1 + \dfrac{5}{100}\right)^4 = 2431$

Example 2:

Cyrus earned S$563 after placing S$3, 000 in a bank for 5 years that offers interest compounded annually. What is the bank's interest rate?

– $563 = 3000\left(1 + \dfrac{r}{100}\right)^5 - 3000$

$\dfrac{563 + 3000}{3000} = \left(1 + \dfrac{r}{100}\right)^5$

$\sqrt[5]{1.188} = 1 + \dfrac{r}{100}$

$r = 100(1.035 - 1) = 3.5$

– The bank's interest rate was 3.5%.

Standard Form

The standard form represents large or small numbers compactly using powers of ten.

$$\underbrace{100 ... 00}_{m}.0 = 1.0 \times 10^m$$

$$0.\underbrace{00 ... 001}_{m} = 1.0 \times 10^{-m}$$

Example 1:

Express the following numbers in standard form.

a) 351, 120, 900 b) −0.000105

a) – 351120900

$= 3.51 \times 100000000$

$= 3.51 \times 10^8$

b) – − 0.000105

$= \dfrac{-1.05}{10000}$

$= -1.05 \times 10^{-4}$

SI Prefixes

If the standard form reminded you of SI units, you were spot on!

Standard Form	Common Name	SI Prefix	Symbol	Example
10^{12}	trillion	tera-	T	terabyte
10^9	billion	giga-	G	gigahertz
10^6	million	mega-	M	megapixel
10^3	thousand	kilo-	k	kilogram
10^{-3}	thousandth	milli-	m	milliliter
10^{-6}	millionth	micro-	μ	micrometer
10^{-9}	billionth	nano-	n	nanosecond
10^{-12}	trillionth	pico-	p	picofarad

Example 1:

The orbit of Halley's Comet around the sun is highly eccentric. Its closest point is 88 million km from the sun, while its farthest distance is 5.2 billion km. Find the ratio between the shortest and farthest distances in standard form.

– $88 \times 10^6 \times 10^3 \ : \ 5.2 \times 10^9 \times 10^3$

$88 \ : \ 5.2 \times 10^3$

$1 \ : \ 5.91 \times 10^1$

Example 2:

A submarine travels at a constant speed from Port P to Port Q. During the journey, it passes Island I in 7 hours. The distance from Port P to Island I is 4.2×10^4 meters.

a) Find the distance the submarine travels in 24 hours. Present your answer in standard form.

b) Given that the distance between Port P and Port Q is 1.08×10^7 meters, determine how long (in days) the journey will take.

a) – Let s = Speed of submarine

t_{PI} = Time taken to travel from P to I

d_7 = Distance traveled in 7 hours

– $s = \dfrac{d_7}{t_{PI}} = \dfrac{4.2 \times 10^4}{7} = 6.0 \times 10^3$

– $d_{24} = s \times 24$

$= 6.0 \times 10^3 \times 24$

$= 1.44 \times 10^5$ meters

b) – $t_{PQ} = \dfrac{d_{PQ}}{s}$
$= \dfrac{1.08 \times 10^7}{6.0 \times 10^3}$
$= 1.8 \times 10^2$
$= 180$ hours
$= 7$ days 12 hours

Glossary

What's the difference between

1. an expression, a function, an equation, and a term?

Expression: A combination of numbers, variables, and operations without equality or inequality ($=, >, <, \leq, \geq, \neq$) signs.

$$x^2 + 2x + 1$$

Function: An expression with a "label" that defines the input (often x).

$$f(x) = x^2 + 2x + 1$$

Equation: A statement where two expressions are equal.

$$x^2 + 2x + 1 = 4x - 1$$

Term: A single part of an expression, separated by $+$ or $-$ signs.

$$2x$$

2. a solution, a root, an intercept, and an intersect?

Solution: A value of the variable that makes an equation true.

$x = -2$ solves $2x + 1 = -3$

Root: A solution when an expression (or function) is set equal to zero.

$x = -2$ is a root of $2x + 4$
as it solves $2x + 4 = 0$

Intercept: A point where a graph crosses the x-axis or y-axis.

$y = 2x + 4$ intercepts the x-axis at $x = -2$
and the y-axis at $y = 4$

Intersect: A point where two curves (or lines) meet.

$y = 2x + 1$ intersects $y = -3$ at $x = -2$

3. a variable and a constant?

The difference depends entirely on the context.

Variable: Value that can vary within the context of a specific problem.
Constant: Value that is fixed within the context of a specific problem.

Indeed x is a variable while a, b, and c are constants in

$$f(x) = ax^2 + bx + c$$

However, x is a constant while θ_0, θ_1, and θ_2 are variables in

$$f(\theta) = \theta_2 x^2 + \theta_1 x + \theta_0$$

In the first example, we are thinking in terms of a quadratic equation and trying to solve for the variable x given constants a, b, and c. In the second example, we are thinking in terms of a linear model (not covered in syllabus) and trying to find the optimal values for variables θ_0, θ_1, and θ_2 given constant x observed through (already collected) data.

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