# Math Hawker Digest — Secondary 3

# Elementary Mathematics – Geometry part II

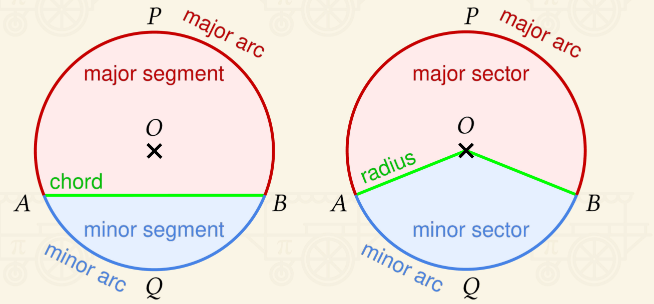
**1. Circle Measurements** **2. Circle Geometry** **3. Other Shapes**

## 1 Circle Measurements

Circles–everyone's favourite shape! Let's start with some basics.

### Circle Anatomy

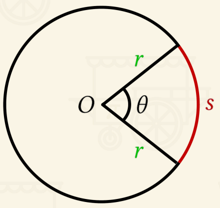
What we call each part of a circle depends on how we divide it.




We name chords using their endpoints. For arcs, segments, and sectors, we use three or four points to tell the major and minor parts apart. Always list the points in order, clockwise or anti-clockwise, to avoid confusion.

### Radian and Arc Length

The radian is defined using arc length. Since a full circle has circumference $2\pi r$, a complete rotation is $2\pi$ radians.




**Arc Length**

- $\text{Angle in Radians} = \dfrac{\text{Arc Length}}{\text{Radius}}$ • $\theta = \dfrac{s}{r}$

**Radian Definition**

- $\text{Angle of Full Circle} = \dfrac{\text{Circumference}}{\text{Radius}}$
- $360^\circ = \dfrac{2\pi r}{r} = 2\pi \text{ rad}$

  $180^\circ = \pi \text{ rad}$

rad is a dimensionless unit that represents radians and directly expresses the ratio $\dfrac{s}{r}$.

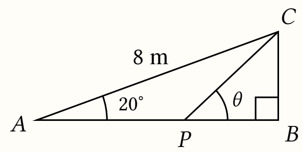
### Circle Formulas

Here is a summary of how the circle is measured in both degrees and radians.

| | Degree | Radian |
|---|---|---|
| **Single Rotation** | $0^\circ \leq \theta \leq 360^\circ$ | $0 \text{ rad} \leq \theta \leq 2\pi \text{ rad}$ |
| **Arc Length** | $\dfrac{\theta}{360^\circ} 2\pi r$ | $r\theta$ |
| **Sector Area** | $\dfrac{\theta}{360^\circ} \pi r^2$ | $\dfrac{1}{2} r^2 \theta$ |

**Example 1:**

In $\triangle ABC$, $CA = 8$ m, $\angle CAB = 20^\circ$, and $\angle ABC = 90^\circ$. Given that $CA : 2CP$, find angle $\theta$ in radians.




a) – $\angle CAB = 20^\circ \times \dfrac{\pi}{180^\circ} = 0.349 \text{ rad}$

– $\sin(0.349) = \dfrac{BC}{8}$

$BC = 2.74$ m

– $CP = \dfrac{8}{2} = 4$ m

– $\sin(\theta) = \dfrac{2.74}{4}$

$\theta = 0.755$ rad

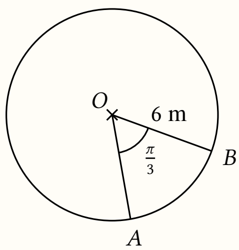
**Example 2:**

In the figure below, a circle with center $O$ has a radius of 6 m. It is divided into sectors with minor angle $\angle AOB = \dfrac{\pi}{3}$ rad.

a) Find the length of minor arc $AB$.

b) Find the area of minor sector $AOB$.

c) Find the major angle in degrees.




a) – $\text{Arc length} = r\theta = 6 \times \dfrac{\pi}{3} = 2\pi$ m

b) – $\text{Sector area} = \dfrac{1}{2} r^2 \theta = \dfrac{1}{2} \times 6^2 \times \dfrac{\pi}{3} = 6\pi \text{ m}^2$

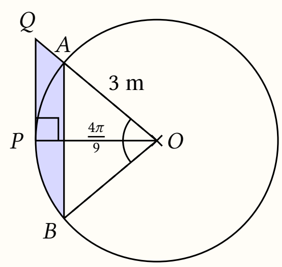
c) – $\text{Major } \angle AOB = \left(2\pi - \dfrac{\pi}{3}\right) \times \dfrac{180^\circ}{\pi} = 300^\circ$

**Example 3:**

In the figure below, a circle with center $O$ has a radius of 3 m. It is divided into sectors with minor angle $\angle AOB = \dfrac{4\pi}{9}$ rad. A tangent $QP$ touches the circle at point $P$, the chord $AB$ forms a minor segment with the arc $APB$, and $OP$ bisects $\angle AOB$.

a) Find the perimeter of region $AQP$.

b) Find the area of shaded region $AQPB$.



a) – $\angle AOP = \frac{4\pi}{9 \times 2} = \frac{2\pi}{9}$ rad

– $QP = 3\tan\left(\frac{2\pi}{9}\right) = 2.52$ m

– $AQ = \frac{3}{\cos\left(\frac{2\pi}{9}\right)} - 3 = 0.916$ m

– Arc $AP = 3 \times \frac{2\pi}{9} = \frac{2\pi}{3}$ m

– $\text{Perimeter}_{AQP} = 2.52 + 0.916 + \frac{2\pi}{3}$
$= 5.53$ m

b) – Let $M$ be the midpoint of chord $AB$.

– $MO = 3\cos\frac{2\pi}{9} = 2.30$ m

– $AM = 3\sin\frac{2\pi}{9} = 1.93$ m

– $\text{Area}_{\triangle MOA} = \frac{1}{2} \times 2.30 \times 1.93 = 2.22 \text{ m}^2$

– $\text{Area}_{AQPB} = \text{Area}_{AQPM} + \text{Area}_{PMB}$

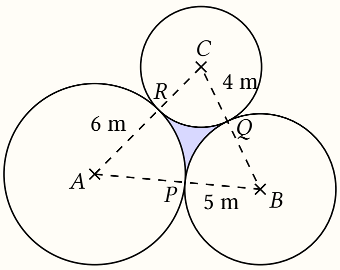
$$= \left(\frac{1}{2} \times 3 \times 2.52 - 2.22\right) + \left(\frac{1}{2} \times 3^2 \times \frac{2\pi}{9} - 2.22\right) = 2.49 \text{ m}^2$$

**Example 4:**

In the figure below, three circles with centers $A$, $B$, and $C$ have radii 6 m, 5 m, and 4 m, respectively. The circle with center $A$ touches two other circles with centers $B$ and $C$ at points $P$ and $R$, respectively. The circles with centers $B$ and $C$ also touch at point $Q$.

a) Find the perimeter of the shaded region $PQR$.

b) Find the area of the shaded region $PQR$.




a) – $9^2 = 10^2 + 11^2 - 2 \times 10 \times 11 \times \sin(\angle CAB)$
$\angle CAB = 0.881$ rad

– $\text{Arc length}_{PR} = 6 \times 0.881 = 5.29$ m

– $10^2 = 9^2 + 11^2 - 2 \times 9 \times 11 \times \sin(\angle ABC)$
$\angle ABC = 1.03$ rad

– $\text{Arc length}_{QP} = 5 \times 1.03 = 5.15$ m

– $\angle BCA = \pi - 0.881 - 1.03 = 1.23$ rad

– $\text{Arc length}_{RQ} = 4 \times 1.23 = 4.92$ m

– $\text{Perimeter}_{PQR} = 5.29 + 5.15 + 4.92 = 15.36$ m

b) – $\text{Sector Area}_{APR} = \frac{1}{2} \times 6^2 \times 0.881 = 15.9 \text{ m}^2$

– $\text{Sector Area}_{BQP} = \frac{1}{2} \times 5^2 \times 1.03 = 12.9 \text{ m}^2$

– $\text{Sector Area}_{CRQ} = \frac{1}{2} \times 4^2 \times 1.23 = 9.85 \text{ m}^2$

– $\text{Area}_{\triangle ABC} = \frac{1}{2} \times 10 \times 11 \times \sin(0.881) = 42.4 \text{ m}^2$

– $\text{Shaded Area}_{PQR} = 42.4 - 15.9 - 12.9 - 9.85$
$= 3.85 \text{ m}^2$

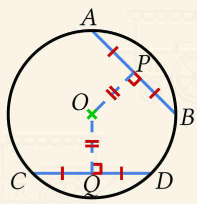
# 2 Circle Geometry

A couple of circles and lines… WCGW?

## Symmetric Properties

The symmetry of circles gives rise to many useful properties.

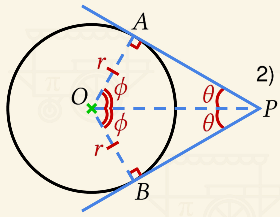
**Chords**




1) Perpendicular bisector of a chord passes through the center $O$.
- Abbreviation: $\perp$ bisector of chord
- Mathematical results:
  - (i) $OP \perp AB \Leftrightarrow OP$ bisects $AB$
  - (ii) $\perp$ bisector of a chord will pass through $O$

2) Equal chords are equally distant from the center $O$.
- Abbreviation: equal chords
- Mathematical results:
  - (i) $OP = OQ \Leftrightarrow AB = CD$

**Tangents**




1) A tangent meets the radius at a right angle.
- Abbreviation: tangent $\perp$ radius
- Mathematical results:
  - (i) $AP \perp OA$

2) Tangents from the same point are equal and form congruent triangles.
- Abbreviation: tangent from ext. point
- Mathematical results:
  - (i) $AP = BP$
  - (ii) $\angle APO = \angle BPO$
  - (iii) $\angle AOP = \angle BOP$

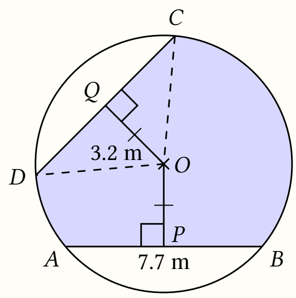
**Example 1:**

In the figure below, a circle with center $O$ is divided into segments. Chord $AB = 7.7$ m and line $OQ = 3.2$ m.

a) Find $CQ$.

b) Find the radius $r$.

c) Find the area of the shaded region $ABCD$.




a) – $CD = AB = 7.7$ m (equal chords)

– $CQ = \frac{1}{2}CD = 3.85$ m ($\perp$ bisector of chord)

b) – $r^2 = 3.85^2 + 3.2^2$
$r = 5.0$ m

c) – $\angle COQ = \cos^{-1}\left(\frac{3.2}{5}\right) = 0.876$ rad

– $\angle COD = 2 \times 0.876 = 1.752$ rad ($\perp$ bisector)

– $\text{Segment Area} = \frac{1}{2} \times 5^2 \times 1.752 - \frac{1}{2} \times 7.7 \times 3.2$
$= 9.59 \text{ m}^2$

– $\text{Shaded Area}_{ABCD} = \pi \times 5^2 - 2 \times 9.59$
$= 59.4 \text{ m}^2$

**Example 2:**

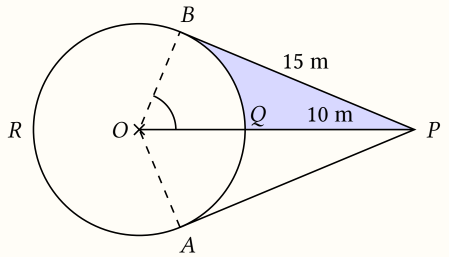
In the figure below, $AP$ and $BP$ are tangent to a circle with center $O$ at points $A$ and $B$, respectively. $BP = 15$ m, $QP = 10$ m, and $OQP$ is a straight line.

a) Find the radius $r$ of the circle.

b) Find the $\angle BOP$.

c) Find the area of the shaded region $QPB$.

d) Find the major arc length $BRA$.




a) – $\angle POB = \frac{\pi}{2}$ (tangent ⊥ radius)

– $15^2 + r^2 = (10 + r)^2$

$r = 6.25$ m

b) – $\angle BOP = \sin^{-1}\left(\frac{15}{10 + 6.25}\right) = 1.176$ rad

c) – Shaded $\text{Area}_{QPB} = \text{Area}_{\triangle OPB} - \text{Sector Area}_{BOQ}$

$= \left(\frac{1}{2} \times 15 \times 6.25\right)$

$- \left(\frac{1}{2} \times 6.25^2 \times 1.176\right)$

$= 23.9 \text{ m}^2$

d) – $\angle QOA = \angle BOQ$ (tangent from ext. point)

$\text{Arc Length}_{BRA} = 6.25 \times (2\pi - 2 \times 1.176)$

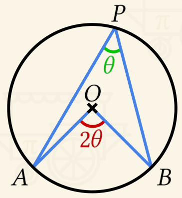
$= 24.6$ m

## Angle Properties

These rules dictate how angles and arcs interact, providing essential tools for solving circle-related problems and understanding their symmetry.

**Centre**

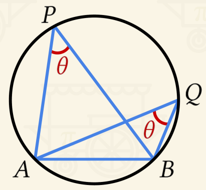
∠ at ctr. = 2∠ at circ.




$\angle AOB = 2\angle APB$

**Same Segment**

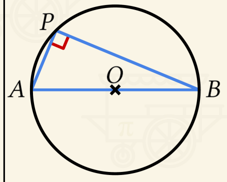
∠s in same segment




$\angle APB = \angle AQB$

**Semicircle**

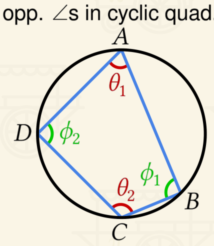
rt. ∠ in semicircle




$\angle APB = 90^\circ$

**Cyclic Quadrilateral**

opp. ∠s in cyclic quad.

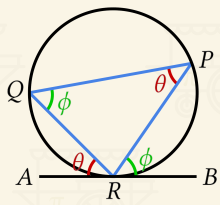



$\angle DAB + \angle BCD = 180^\circ$

$\angle DAB + \angle BCD = 180^\circ$

**Tangent-Chord Theorem**

tan-chord theorem

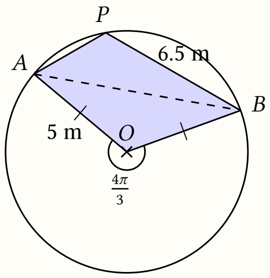



$\angle ARQ = \angle RPQ$

$\angle PRB = \angle PQR$

**Example 1:**

In the figure below, a circle with center $O$ has a radius of 5 m. Given that $BP = 6.5$ m and obtuse $\angle AOB = \frac{4\pi}{3}$ rad, find the area of the quadrilateral $AOBP$.




– Acute $\angle AOB = 2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$ rad

– $\text{Area}_{\triangle AOB} = \frac{1}{2} \times 5 \times 5 \times \sin\left(\frac{2\pi}{3}\right) = 10.8 \text{ m}^2$

– $\angle APB = \frac{1}{2} \times \frac{4\pi}{3} = \frac{2\pi}{3}$ rad (half of center ∠)

– $AB^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos\frac{2\pi}{3}$

$AB = 8.66$ m

– $\frac{8.66}{\sin\left(\frac{2\pi}{3}\right)} = \frac{6.5}{\sin(\angle PAB)}$

$\angle PAB = 0.708$ rad

– $\angle ABP = \pi - \frac{2\pi}{3} - 0.708 = 0.340$ rad

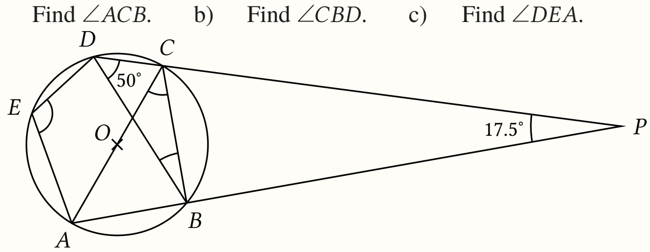
– $\text{Area}_{\triangle ABP} = \frac{1}{2} \times 6.5 \times 8.66 \times \sin(0.340) = 9.38 \text{ m}^2$

– $\text{Area}_{AOBP} = 10.8 + 9.38 = 20.2 \text{ m}^2$

**Example 2:**

In the figure below, a circle with center $O$ has points $A$, $B$, $C$, $D$, and $E$ on its circumference. $AOC$, $ABP$, and $PCD$ are straight lines, $\angle CDB = 50^\circ$, and $\angle BPC = 17.5^\circ$.

a) Find $\angle ACB$. b) Find $\angle CBD$. c) Find $\angle DEA$.




a) – $\angle CAB = \angle CDB = 50^\circ$ (∠s in same segment)

– $\angle ABC = 90^\circ$ (rt. ∠ in semicircle)

– $\angle ACB = 180 - 90 - 50 = 40^\circ$

b) – $\angle PBC = 180 - 90 = 90^\circ$

– $\angle BCP = 180 - 90 - 17.5 = 72.5^\circ$

– $\angle ACD = 180 - 40 - 72.5 = 67.5^\circ$

– $\angle CBD = 180 - 50 - (67.5 + 40) = 22.5^\circ$

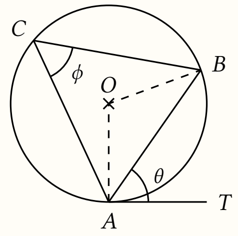
c) – $ACDE$ is a cyclic quadrilateral.

⇒ $\angle DEA = 180 - 67.5$ (opp. ∠s in cyc. quad.)

$= 112.5^\circ$

**Example 3:**

Prove the tangent-chord theorem by showing that $\theta = \phi$.




– $OA = OB$ (radii)

⇒ $ABO$ is an isos. △.

⇒ $\angle AOB = 180^\circ - \angle OAB - \angle ABO$

$= 180^\circ - 2\angle OAB$

– $\angle OAT = 90^\circ$ (tangent ⊥ radius)
⇒ $\angle OAB = 90^\circ - \theta$
⇒ $\angle AOB = 180^\circ - 2(90^\circ - \theta) = 2\theta$
– $\angle BCA = \frac{2\theta}{2} = \theta$ (half of center ∠)
⇒ $\theta = \phi$

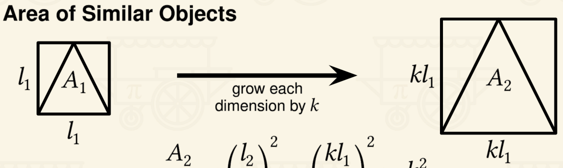
# 3 Other Shapes

Great job so far! Now let's put together everything we've learnt.

## 3D Shapes and Similarity Ratios

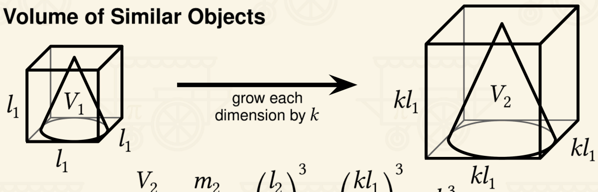
Similarity ratios apply to more than just lengths of similar triangles. The areas and volumes of similar shapes scale with the square and cube of the length ratio, respectively.

**Area of Similar Objects**




$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2 = \left(\frac{kl_1}{l_1}\right)^2 = k^2$$

**Volume of Similar Objects**




$$\frac{V_2}{V_1} = \frac{m_2}{m_1} = \left(\frac{l_2}{l_1}\right)^3 = \left(\frac{kl_1}{l_1}\right)^3 = k^3$$

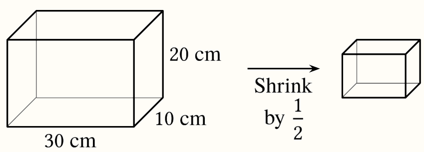
where $l_i$, $A_i$, $V_i$, and $m_i$ are the length, area, volume, and mass of object $i$.

**Example 1:**

A company produces gift boxes in the shape of cuboids. The original design has dimensions 30 cm × 20 cm × 10 cm. Due to customer demand for a smaller version, the company wants to create a scaled-down version of the box where all dimensions are reduced in the same ratio.

a) If the new box is half the height of the original, calculate the volume of the smaller box.

b) If the company now wants to double the surface area of the smaller box to make a more visually appealing design, by what factor should they scale up its dimensions?




a) – Let $V_1$ = Original Volume
$V_2$ = New Volume
– $V_1 = 30 \times 20 \times 10 = 6000 \text{ cm}^3$
– $\frac{V_2}{V_1} = \left(\frac{1}{2}\right)^3$
$V_2 = \frac{6000}{8} = 750 \text{ cm}^3$

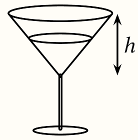
b) – Let $A_1$ = Original Surface Area
$A_2$ = New Surface Area
– $\frac{A_2}{A_1} = 2 = (\sqrt{2})^2$
∴ They should scale up by a factor of $\sqrt{2}$.

**Example 2:**

Bartender Anton serves martinis in 8-ounce conical glasses. The cost of ingredients is \$1 per ounce, and he sells each drink for \$15. He fills each glass to 80% of its full height.

a) What percentage of the total cocktail volume is saved by not filling to the brim?

b) If Anton fills the glasses to 90% instead of 80%, how much money does he lose per glass due to this change?




a) – Let $V_1$ = Full Volume
$V_2$ = Served Volume
– $\frac{V_2}{V_1} = \left(\frac{80}{100}\right)^3 = 0.512$
– Percentage Saved = $1 - 0.512 = 0.488 = 48.8\%$

b) – Cost Increase = $\left[\left(\frac{90}{100}\right)^3 - \left(\frac{80}{100}\right)^3\right] \times 8 \times 1 = \$1.74$