Math Hawker Digest — Secondary 3

Elementary Mathematics – Graphs

1. Coordinate Geometry **2. Quadratic Graphs** **3. More Graphs**

1 Coordinate Geometry

What happens when algebra meets geometry on a flat plane? French philosopher René Descartes has the answer for you!

Coordinate Geometry Basics

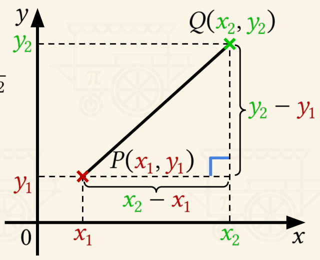
Coordinates $P(x, y)$ mark a point P's position along the x- and y-axes. Two points form a line segment and its length can be calculated using the Pythagorean theorem. The gradient tells us how steep the line segment is.

Length

- $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Gradient

- $m_{PQ} = \dfrac{y_2 - y_1}{x_2 - x_1}$



Equation of a Line

A line consists of infinitely many points, and its equation is a rule that every point on the line satisfies. A line's equation connects its slope and y-intercept to any point that lies on it.

coordinates

$$y = mx + c$$

gradient y-intercept

Example 1:

Given points $P(-3, 1)$ and $Q(5, -6)$, find the length and gradient of line segment PQ.

- $PQ = \sqrt{[5-(-3)]^2 + [-6-1]^2} = \sqrt{113} = 10.64$ units
- $m_{PQ} = \dfrac{-6-1}{5-(-3)} = \dfrac{-7}{8} = 0.875$

Example 2:

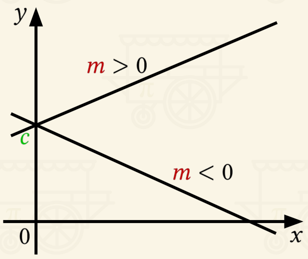
Points P and Q have coordinates $(4, -3)$ and $(2, 1)$, respectively, and point Z lies on the x-axis. Find the possible coordinates of Z such that $PZ = 2QZ$.

- $PZ = \sqrt{[x-4]^2 + [0-(-3)]^2}$
  $= \sqrt{(x-4)^2 + 9}$
- $QZ = \sqrt{(x-2)^2 + (0-1)^2}$
  $= \sqrt{(x-2)^2 + 1}$
- Since $PZ = 2QZ$,
  $\sqrt{(x-4)^2 + 9} = 2\sqrt{(x-2)^2 + 1}$
  $(x-4)^2 + 9 = 2^2[(x-2)^2 + 1]$
  $x^2 - 8x + 16 + 9 = 4(x^2 - 4x + 4 + 1)$
  $-3x^2 + 8x + 5 = 0$
- $x = \dfrac{-8 \pm \sqrt{8^2 - 4(-3)(5)}}{2(-3)}$
  $= \dfrac{4}{3} \pm \dfrac{\sqrt{31}}{3}$
  $x = 3.19$ or $x = -0.523$
- The possible coordinates of Z are $(3.19, 0)$ and $(-0.523, 0)$.

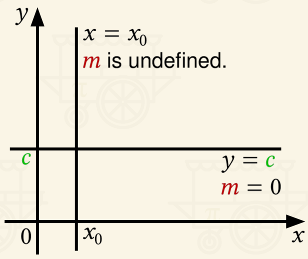
Gradients and Lines

Gradients dictate the direction of lines.

Positive/Negative Gradient



Horizontal/Vertical Line



Example 1:

A line crosses the x-axis at $x = 3$ and the y-axis at $y = -2$. Determine its equation.

- $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-2-0}{0-3} = \dfrac{2}{3}$
- $y = \dfrac{2}{3}x - 2$

Example 2:

Given points $P(6, 4)$, $Q(x, -7)$, and $R(2, 4x)$, find the possible values of x if the gradient of PQ equals the gradient of QR.

- $m_{PQ} = m_{QR}$
  $\dfrac{-7-4}{x-6} = \dfrac{4x-(-7)}{2-x}$
  $-11(2-x) = (4x+7)(x-6)$
  $4x^2 - 28x - 20 = 0$
  $x^2 - 7x - 5 = 0$
- $x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-5)}}{2(1)}$
  $= \dfrac{7}{2} \pm \dfrac{\sqrt{69}}{2}$
  $x = 7.65$ or $x = -0.653$

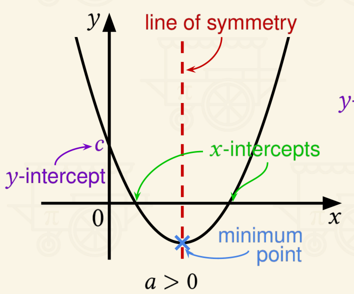
2 Quadratic Graphs

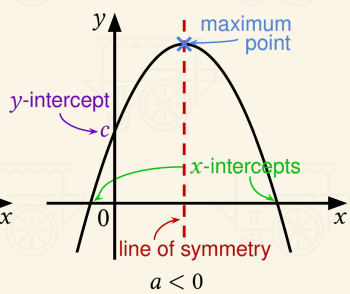
Let's explore a visual way of understanding quadratic equations!

Quadratic Graph Anatomy

The quadratic graph is a parabola (symmetric "u" or "n" shape). It always has 1 y-intercept and can have 0, 1, or 2 x-intercepts.

$$y = ax^2 + bx + c$$





Example 1:

The table below shows some values of x and the corresponding values of y, where $y = -x^2 + 2x + 1$.

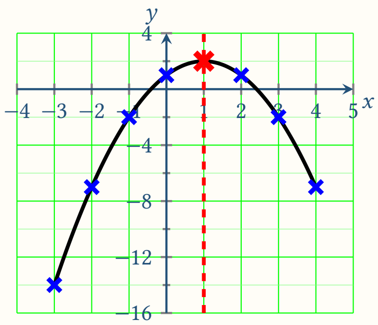
x	-3	-2	-1	0	1	2	3	4
y	-14			1				-7

a) Complete the table.

b) Using a suitable scale, plot the points in the table and join them with a smooth curve. Mark the line of symmetry.

c) Use the graph to find the x-intercepts and the coordinate of the maximum point.

x	-3	-2	-1	0	1	2	3	4
y	-14	-7	-2	1	2	1	-2	-7



c) – $x \approx 0.4$ and $x \approx 2.4$

– Maximum point is $(1, 2)$.

Example 2:

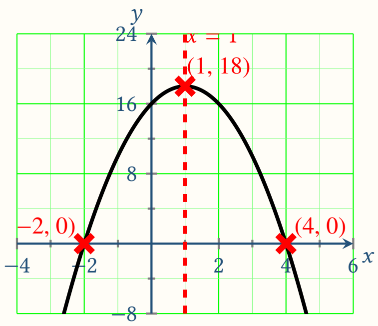
a) Factorize $y = -2x^2 + 4x + 16$.

b) Hence, sketch the graph, indicating all axes intercepts, turning points, and lines of symmetry.

a) – $y = -2x^2 + 4x + 16$

$= -2(x^2 - 2x - 8)$

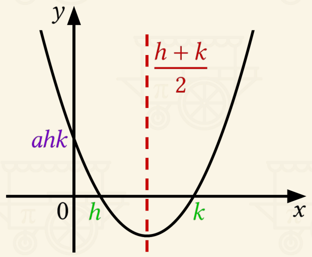
$= -2(x - 4)(x + 2)$



Alternate Forms

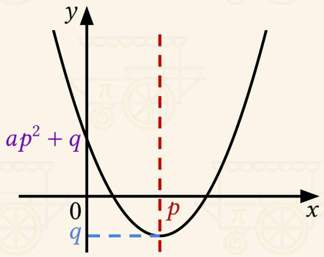
Factorized and completed square forms of quadratic expressions help directly identify key features of the graph.

$y = a(x - h)(x - k)$



x-intercepts always exist if the quadratic expression is factorizable, thus this form always has 2 x-intercepts.

$y = a(x - p)^2 + q$



This form does not show the x-intercepts directly. They only exist if the curve crosses the x-axis, which happens when:

1) $a > 0$ and $q \leq 0$
2) $a < 0$ and $q \geq 0$

Example 1:

A quadratic curve has a turning point at $(7, -3)$ and a y-intercept at $y = 95$. Find its x-intercepts.

– $y = ap^2 + q$

$95 = a \times 7^2 - 3$

$a = 2$

– $y = a(x - p)^2 + q$

$0 = 2(x - 7)^2 - 3$

$x = 7 \pm \sqrt{\frac{3}{2}}$

Example 2:

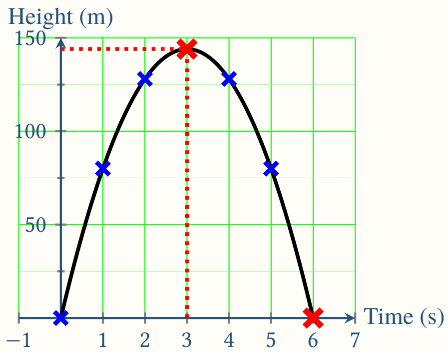
An arrow is shot straight up from ground level with an initial velocity of 96 m/s. Its height y after time t of being shot can be modeled by $y = -16t^2 + 96t$.

a) Using a suitable scale, plot the graph of $y = -16t^2 + 96t$ for $0 \leq t \leq 6$.

b) Use the graph to find when the arrow hits the ground.

c) Use the graph to find the arrow's maximum height and time taken to reach said height.

x	0	1	2	3	4	5	6
y	0	80	128	144	128	80	0



b) – Arrow hits the ground at $t = 6$ s.

c) – Max height is at $y = 144$ m.

– Time taken is $t = 3$ s.

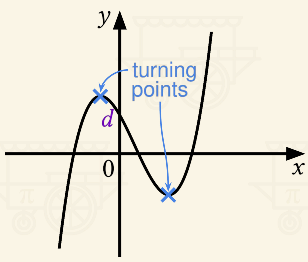
3 More Graphs

More fancy curves and their properties!

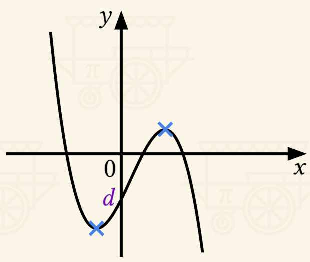
Cubic Graph

Cubic graphs are the next step after quadratic graphs and can have 2 turning points.

$y = ax^3 + bx^2 + cx + d$



$a > 0$



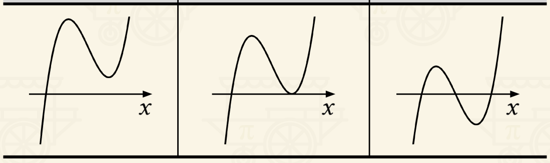
$a < 0$

Turning points are points where the curve changes direction. We call them turning points here instead of the usual minimum or maximum points because the curves extend beyond them.

Intercepts of Cubic Graphs

Cubic graphs always have 1 y-intercept and can have 1, 2, or 3 x-intercepts.

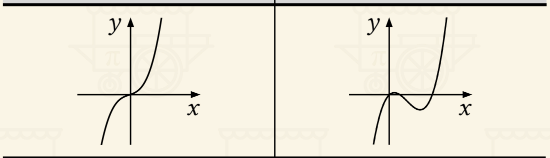
Case 1: 1 x-intercept	Case 2: 2 x-intercepts	Case 3: 3 x-intercepts



Turning Points of Cubic Graphs

Determining the number of turning points of cubic graphs is generally not straightforward, but there are two cases.

Case 1: 0 turning points	Case 2: 0 or 2 turning points



$y = +x^3 + x^2 + x$	$y = +x^3 - 3x^2 + x$
All coefficients have the same signs.	Coefficients have different signs.

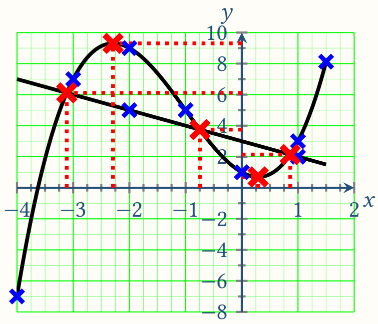
Example 1:

Using a suitable scale, plot the graph of $y = x^3 + 3x^2 - 2x + 1$ for $-4 \leq x \leq 1.5$.

a) Use your graph to find the coordinates of the turning points.

b) By drawing a line on the same axes, solve $x^3 + 3x^2 - x - 2 = 0$.

x	-4	-3	-2	-1	0	1	1.5
y	-7	7	9	5	1	3	8.125



a) – Turning points are $(-2.3, 9.3)$ and $(0.3, 0.7)$.

b) – $x^3 + 3x^2 - x - 2 = 0$

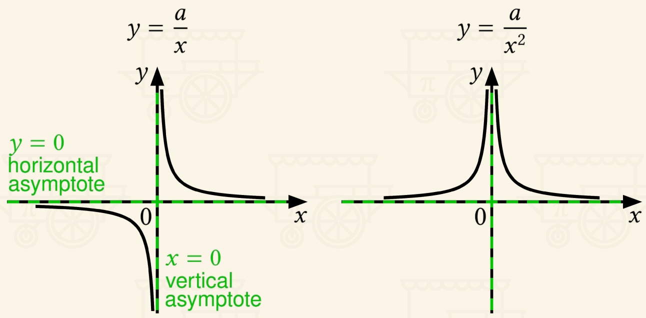
$x^3 + 3x^2 - 2x + 1 = -x + 3$

x	-2	1
y	5	2

– $x \approx -3.1$, $x \approx -0.75$, or $x \approx 0.9$

Reciprocal Curves

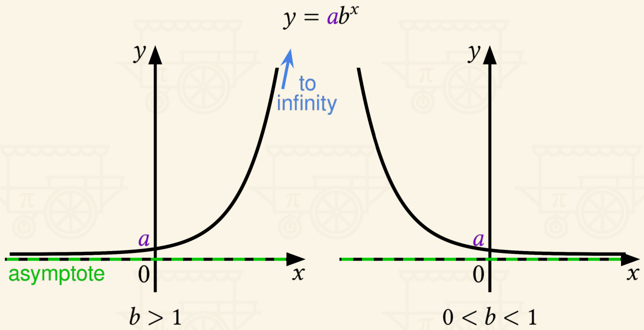
Reciprocal graphs feature vertical and horizontal asymptotes—lines the curve approaches but never touches. The x- and y-axes are the asymptotes in a basic reciprocal curve.



These graphs show the case when $a > 0$. For $a < 0$, the graphs are reflected vertically.

Exponential Curves

Exponential curves are always monotonic (ever-increasing or ever-decreasing) and approach a horizontal asymptote (x-axis) with no turning points.



Again, these graphs show the case when $a > 0$.

Example 1:

The table below shows some values of x and the corresponding values of y, where $y = \frac{1}{x^2}$.

x	-4	-2	-1	-0.5	0.5	1	2	4
y		0.25				1.00		

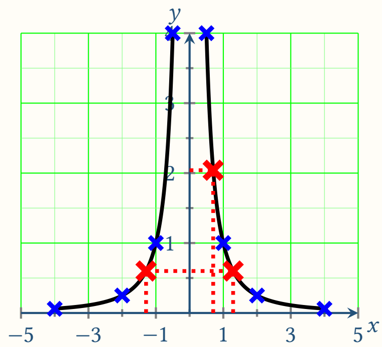
a) Complete the table, leaving your answers to 2 decimal places.

b) Using a suitable scale, plot the points in the table and join them with a smooth curve.

c) Use the graph to find y when $x = 0.7$.

d) Use the graph to find the values of x when $y = 0.6$.

x	-4	-2	-1	-0.5	0.5	1	2	4
y	0.06	0.25	1.00	4.00	4.00	1.00	0.25	0.06

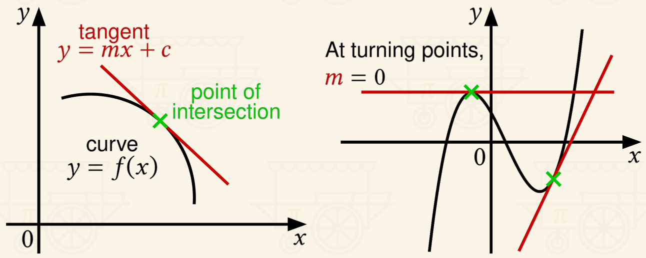


c) – $y \approx 2$

d) – $x \approx \pm 1.3$

Tangents

Tangents are lines that touch a curve at exactly one point, representing the curve's gradient at that point.



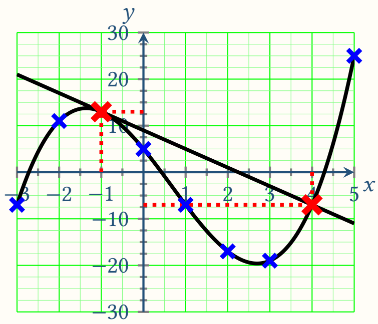
Example 1:

Using a suitable scale, plot the graph of $y = x^3 - 2x^2 - 11x + 5$ for $-3 \le x \le 5$.

a) By drawing a tangent, find the gradient of the curve at $x = -1$.

b) Use the graph to find the second intersection point between the tangent and the curve.

x	-3	-2	-1	0	1	2	3	4	5
y	-7	11	13	5	-7	-17	-19	-7	25

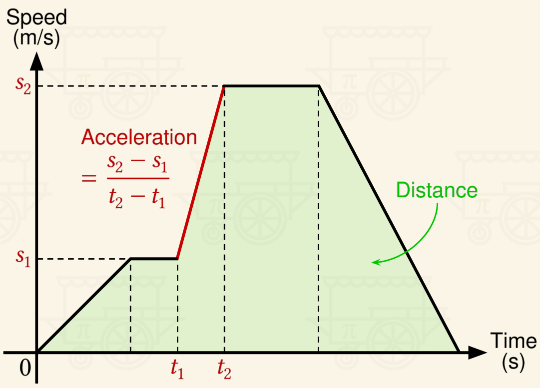


a) – $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13 - 5}{-1 - 1} = -4$

b) – Intersection point is at $(1, -7)$.

Applications of Graphs

Curves model real-world phenomena. Tangents and gradients represent rates of change, while areas under curves represent accumulated quantities like distance, energy, or profit.



Example 1:

The graph below shows the speed-time graph of a car.

The car accelerated to the speed limit before remaining at a constant speed during a traffic jam for 2 hours.

a) Determine the speed limit and the car's speed during the jam.

b) Calculate the car's average speed for the entire journey.

a) – The speed limit is 90 km/h and the car's speed during the jam was 30 km/h.

b) – Total Distance

$$= \frac{1}{2} \times 1 \times 90 + 90 \times 2 + 30 \times 2 + \left(30 \times 2 + \frac{1}{2} \times 2 \times 20\right) + \frac{1}{2} \times 4 \times 50$$
$$= 45 + 180 + 60 + 60 + 20 + 100$$
$$= 465 \text{ km}$$

– Average Speed $= \frac{\text{Total Distance}}{\text{Total Time}} = \frac{465}{11} = 42.3$ km/h

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