Math Hawker Digest **Secondary 4**

Additional Mathematics – Differentiation

1. Fundamentals
2. Elementary Rules
3. Identities
4. Graph Applications

1 Fundamentals

Let's start with some basic understanding of differentiation.

Derivative Definition

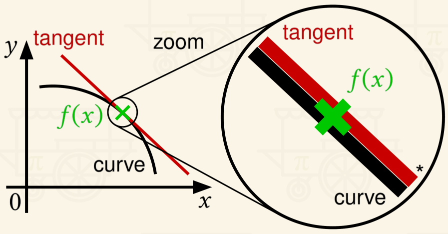
Differentiation is a fundamental tool for finding how quickly a function is changing at any given point. The result of this process is called the **derivative**, written as $f'(x)$ (read as "f prime x"), which tells us the rate of change of the function $f(x)$ at a given value of x.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

where $\lim_{h\to 0}$ means "as h becomes really small (approaching 0)", and h represents a small horizontal change. Don't worry about this new limit notation, just understand it conceptually. You won't need it rigorously until university-level calculus.

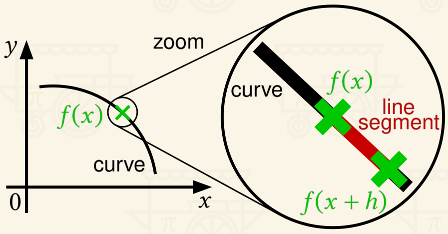
Derivative Intuition

Derivatives are closely related to **gradients**. In multidimensional calculus (beyond the syllabus), the terms "derivatives" and "gradients" can mean slightly differing things; but for now, you can treat them as the same. To understand how differentiation gives us the gradient of a curve at a point, let's first look deeper into how tangents do so.



at small scales, the curve is approximately **linear** and shares the same direction as the tangent, so their gradients match

Instead of drawing tangents all over a curve, the derivative helps us determine the gradient anywhere on the curve.



similarly, the line segment between $f(x)$ and $f(x+h)$ is approximately linear as h gets smaller, behaving like a tangent

*in the first diagram, the tangent and curve should really be overlapping, but it's hopefully more intuitive this way.

Notation and History

Let's briefly introduce the two main notations for derivatives, along with their history and some associated controversies.

Lagrange's Notation

The derivative we defined earlier is written in Lagrange's notation:

$$f'(x)$$

This was introduced by Joseph-Louis Lagrange in the 1700s. His approach emphasized functions and algebraic manipulation, and it fits well with the modern limit-based definition of the derivative, formalized in the 1800s:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Leibniz's Notation

An alternative, widely used notation is:

$$\frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}[f(x)]$$

This was introduced by Gottfried Wilhelm Leibniz in the late 1600s. His calculus relied on infinitesimals — quantities like dy and dx that were "infinitely small" but not zero.

Notation Controversy

- **Is dy/dx a fraction?** Historically, Leibniz treated it as a true ratio of infinitesimals. But after the rise of limit-based calculus in the 1800s, mathematicians stopped treating dy and dx as real quantities. In modern standard calculus, dy/dx is therefore **not** a real fraction. It is simply a symbolic shorthand for the derivative.

  Ironically, infinitesimals were rigorously reintroduced in the 1960s by Abraham Robinson through a new framework called non-standard analysis which makes dy/dx a real fraction within that framework. While logically valid, that framework is not interchangeable with standard calculus, so you should still treat dy/dx as a symbol, not as a fraction, within our syllabus.
- **Why use Leibniz's notation at all?** Despite its complications, Leibniz's notation is incredibly useful. At the introductory level, it can make certain concepts in differentiation more intuitive to understand and apply.

Understanding the Notation

Let's take a closer look at how to interpret the various notations.

Given a function $y = f(x)$, where $f(x)$ is any function of x, all the following notations represent the same idea: the derivative of $f(x)$ with respect to x.

Lagrange's Notation.

$$f'(x) = \underbrace{\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)]}_{\text{Leibniz's Notation}}$$

Emphasizes change between 2 variables.

Emphasizes the function f itself.

Operator-style notation. Treated as an action being applied to a function.

These are 4 ways of representing the derivative and are mathematically equivalent.

2

Elementary Rules

Some basic rules to help with differential calculus.

Power Rule

The power rule is the cornerstone of differentiation. Most other rules build on or extend its idea.

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \frac{d}{dx}[k] = 0$$

Example 1:

For the following functions, find the derivative with respect to x.

a) $y = x^5$ b) $f(x) = \dfrac{1}{x^2}$

c) $f(x) = 2$ d) $y = \sqrt{x^3}$

a) – $y = x^5$

$$\frac{dy}{dx} = 5x^{5-1} = 5x^4$$

b) – $f(x) = \dfrac{1}{x^2}$

$$= x^{-2}$$

$$f'(x) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

c) – $f(x) = 2$

$$f'(x) = 0$$

d) – $y = \sqrt{x^3}$

$$= x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

Scalar Multiple Rule

Constant scalar multiples are unaffected by differentiation and can be factored out of the derivative.

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$$

Addition/Subtraction Rule

The derivative of a sum is the sum of the derivatives of its terms.

$$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$$

where u and v are functions of x.

Example 1:

For the following functions, find the derivative with respect to x.

a) $y = 3x^2 - 4x + 2\sqrt{x}$ b) $y = \dfrac{4x^3 - 2x + a}{x}$

a) – $y = 3x^2 - 4x + 2\sqrt{x}$

$$\frac{dy}{dx} = 3\frac{d}{dx}[x^2] - 4\frac{d}{dx}[x] + 2\frac{d}{dx}[x^{\frac{1}{2}}]$$

$$= 3(2x) - 4(1) + 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= 6x - 4 + \frac{1}{\sqrt{x}}$$

b) – $y = \dfrac{4x^3 - 2x + a}{x}$

$$= 4x^2 - 2 + ax^{-1}$$

$$\frac{dy}{dx} = 4\frac{d}{dx}[x^2] - \frac{d}{dx}[2] + a\frac{d}{dx}[x^{-1}]$$

$$= 4(2x) - (0) + a(-x^{-2})$$

$$= 8x - \frac{a}{x^2}$$

Example 2:

A company models its profit (in thousands of dollars) from with the following function:

$$P(t) = 2t^3 - 15t^2 + 36t$$

where t is the number of months since launch.

a) Find the rate at which profit changes over time.

b) Find the equation of the tangent at $t = 2$.

c) Find the rate of increase in profit the company makes at $t = 3.5$.

a) – $P(t) = 2t^3 - 15t^2 + 36t$

$$P'(t) = 6t^2 - 30t + 36$$

b) – $P'(2) = 6(2)^2 - 30(2) + 36 = 0$

$$P(2) = 2(2)^3 - 15(2)^2 + 36(2) = 28$$

– $y = 0x + 28$

$$y = 28$$

c) – $P'(3.5) = 6(3.5)^2 - 30(3.5) + 36 = 4.5$

The rate of increase is \$4500 per month.

Chain Rule

Some functions are easier to differentiate by recognizing them as composite functions (one function inside another). The chain rule helps us handle these by breaking the process into manageable steps.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y can be simply expressed in terms of u and u can be expressed in terms of x.

Example 1:

For the following functions, find the derivative with respect to x.

a) $y = (x^3 - 2x)^5$ b) $y = \dfrac{1}{\sqrt{x^2 + 1}}$

a) – Let $u = x^3 - 2x$.

$$y = (x^3 - 2x)^5 = u^5$$

– $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{d}{du}[u^5] \times \frac{d}{dx}[x^3 - 2x]$$

$$= 5u^4 \times (3x^2 - 2)$$

$$= 5(x^3 - 2x)^4 \times (3x^2 - 2)$$

$$= 5(3x^2 - 2)(x^3 - 2x)^4$$

b) – Let $u = x^2 + 1$.

$$y = \frac{1}{\sqrt{x^2 + 1}} = u^{-\frac{1}{2}}$$

– $$\frac{dy}{dx} = \frac{d}{du}[u^{-\frac{1}{2}}] \times \frac{d}{dx}[x^2 + 1]$$

$$= -\frac{1}{2}u^{-\frac{3}{2}} \times 2x$$

$$= -x(x^2 + 1)^{-\frac{3}{2}}$$

$$= -\frac{x}{\sqrt{(x^2 + 1)^3}}$$

Product Rule

To find the derivative of a product, we differentiate each function one at a time, keeping the other fixed, and add the two results.

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx}$$

where u and v are functions of x.

Example 1:

For the following functions, find the derivative with respect to x.

a) $y = (2x^3 + x)(3x^2 - 5)$ b) $y = (x^2 - 2)^3\sqrt{2x+1}$

a) – $y = (2x^3 + x)(3x^2 - 5)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[2x^3 + x](3x^2 - 5) + (2x^3 + x)\frac{d}{dx}[3x^2 - 5] \\ &= (6x^2 + 1)(3x^2 - 5) + (2x^3 + x)(6x) \\ &= (6x^2 + 1)(3x^2 - 5) + 6x(2x^3 + x)\end{aligned}$$

b) – $y = (x^2 - 2)^3\sqrt{2x+1}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(x^2 - 2)^3]\sqrt{2x+1} + (x^2 - 2)^3\frac{d}{dx}[\sqrt{2x+1}] \\ &= [3(x^2 - 2)^2(2x)]\sqrt{2x+1} + (x^2 - 2)^3\left[\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)\right] \\ &= 6x(x^2 - 2)^2\sqrt{2x+1} + \frac{(x^2 - 2)^3}{\sqrt{2x+1}}\end{aligned}$$

Example 2:

Differentiate $y = (x^3 + 1)(x^2 + 2)(x + 3)$ with respect to x.

– $y = (x^3 + 1)(x^2 + 2)(x + 3)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(x^3 + 1)](x^2 + 2)(x + 3) \\ &\quad + (x^3 + 1)\frac{d}{dx}[(x^2 + 2)(x + 3)] \\ &= \frac{d}{dx}[(x^3 + 1)](x^2 + 2)(x + 3) \\ &\quad + (x^3 + 1)\left[\frac{d}{dx}[(x^2 + 2)](x + 3) + (x^2 + 2)\frac{d}{dx}[(x + 3)]\right] \\ &= \frac{d}{dx}[(x^3 + 1)](x^2 + 2)(x + 3) \\ &\quad + (x^3 + 1)\frac{d}{dx}[(x^2 + 2)](x + 3) \\ &\quad + (x^3 + 1)(x^2 + 2)\frac{d}{dx}[(x + 3)] \\ &= (3x^2)(x^2 + 2)(x + 3) \\ &\quad + (x^3 + 1)(2x)(x + 3) \\ &\quad + (x^3 + 1)(x^2 + 2)(1) \\ &= 3x^2(x^2 + 2)(x + 3) + 2x(x^3 + 1)(x + 3) + (x^3 + 1)(x^2 + 2)\end{aligned}$$

Quotient Rule

The derivative of a quotient follows this convoluted formula.

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

where u and v are functions of x.

Example 1:

Differentiate $y = \dfrac{(x^2 + 1)^2}{x - 4}$ with respect to x.

– $y = \dfrac{(x^2 + 1)^2}{x + 4}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}[(x^2 + 1)^2](x - 4) - (x^2 + 1)^2\frac{d}{dx}[(x - 4)]}{(x - 4)^2} \\ &= \frac{2(x^2 + 1)(2x)(x - 4) - (x^2 + 1)^2(1)}{(x - 4)^2} \\ &= \frac{[4x(x - 4) - (x^2 + 1)](x^2 + 1)}{(x - 4)^2} \\ &= \frac{(4x^2 - 16x - x^2 - 1)(x^2 + 1)}{(x - 4)^2} \\ &= \frac{(3x^2 - 16x - 1)(x^2 + 1)}{(x - 4)^2}\end{aligned}$$

Higher-order Derivatives

So far, we've only covered the first derivative. Differentiation can be repeated to find the rate of change of the rate of change.

		Derivative		
		First	Second	Third
Notation	Lagrange	$f'(x)$	$f''(x)$	$f'''(x)$
	Leibniz	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
Interpretation	Physics	Velocity	Acceleration	Jerk
	Graphs	Gradient	Curvature	Torsion

Similar to dy/dx, think of d^2y/dx^2 and d^3y/dx^3 as whole symbols representing the second and third derivatives, respectively. There are no "powers" or "fractions" involved.

Example 1:

Find the third derivative of $y = x^4 + x^2 - \dfrac{1}{x}$ with respect to x.

– $y = x^4 + x^2 - x^{-1}$

$$\begin{aligned}\frac{dy}{dx} &= 4x^3 + 2x + x^{-2} \\ \frac{d^2y}{dx^2} &= 12x^2 + 2 - 2x^{-3} \\ \frac{d^3y}{dx^3} &= 24x + 6x^{-4} \\ &= 24x + \frac{6}{x^4}\end{aligned}$$

3 Identities

Differentiation can be applied to common named functions too.

Derivative of Trigonometric Functions

Trigonometric functions can also be differentiated using the following standard identities.

- $\dfrac{d}{dx}[\sin(x)] = \cos(x)$
- $\dfrac{d}{dx}[\sin(u)] = \cos(u)\dfrac{du}{dx}$
- $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$
- $\dfrac{d}{dx}[\cos(u)] = -\sin(u)\dfrac{du}{dx}$
- $\dfrac{d}{dx}[\tan(x)] = \sec^2(x)$
- $\dfrac{d}{dx}[\tan(u)] = \sec^2(u)\dfrac{du}{dx}$

where u is a function of x.

Example 1:

For the following functions, find the derivative with respect to x.

a) $y = \sin^2(x)$ b) $y = [2\tan(x) + 1]^3$

c) $y = \cos(x^2 + \pi)$ d) $y = \sin(3x)\cos(2x)$

a) – $y = \sin^2(x)$

$$\frac{dy}{dx} = 2\sin(x)\cos(x)$$

b) – $y = [2\tan(x) + 1]^3$

$$\frac{dy}{dx} = 3[2\tan(x) + 1]^2[2\sec^2(x)]$$

c) – $y = \cos(x^2 + \pi)$

$$\begin{aligned}\frac{dy}{dx} &= -\sin(x^2 + \pi)(2x) \\ &= -2x\sin(x^2 + \pi)\end{aligned}$$

d) – $y = \sin(3x)\cos(2x)$

$$\begin{aligned}\frac{dy}{dx} &= \cos(3x)(3)\cos(2x) + \sin(3x)[-\cos(2x)](2)\\ &= 3\cos(3x)\cos(2x) - 2\sin(3x)\sin(2x)\end{aligned}$$

Derivative of Exponential Functions

Exponential functions are unique in differentiation because they might not change at all.

- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$

where u is a function of x.

Derivative of Logarithmic Functions

As with exponentials, we will focus on the natural logarithm (logarithm with base e). The identities are simpler for these cases.

- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
- $\frac{d}{dx}[\ln(u)] = \frac{1}{u}\frac{du}{dx}$

where u is a function of x.

Example 1:

For the following functions, find the derivative with respect to x.

a) $y = e^{2x^3+x}$

b) $y = \ln(x^2 - 3x + 4)$

a) – $y = e^{x^3+2x}$

$$\begin{aligned}\frac{dy}{dx} &= e^{2x^3+x}(3x^2+2)\\ &= (3x^2+2)e^{2x^3+x}\end{aligned}$$

b) – $y = \ln(x^2 - 3x + 4)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x^2-3x+4}(2x-3)\\ &= \frac{2x-3}{x^2-3x+4}\end{aligned}$$

Example 2:

Let

$$f(x) = \frac{x\ln(x)}{\sin(x)}$$

a) Differentiate $f(x)$ with respect to x.

b) Find the gradient at $x = \frac{\pi}{2}$.

– $f(x) = \frac{x\ln(x)}{\sin(x)}$

$$\begin{aligned}f'(x) &= \frac{[x\ln(x)]'\sin(x) - x\ln(x)[\sin(x)]'}{\sin^2(x)}\\ &= \frac{\left[\ln(x) + \frac{1}{x}(x)\right]\sin(x) - x\ln(x)\cos(x)}{\sin^2(x)}\\ &= \frac{[\ln(x)+1]\sin(x) - x\ln(x)\cos(x)}{\sin^2(x)}\end{aligned}$$

$$\begin{aligned}f'\left(\frac{\pi}{2}\right) &= \frac{\left[\ln\left(\frac{\pi}{2}\right)+1\right]\sin\left(\frac{\pi}{2}\right) - x\ln\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)}{\sin^2\left(\frac{\pi}{2}\right)}\\ &= \frac{\left[\ln\left(\frac{\pi}{2}\right)+1\right](1) - x\ln\left(\frac{\pi}{2}\right)(0)}{(1)^2}\\ &= \ln\left(\frac{\pi}{2}\right)+1\end{aligned}$$

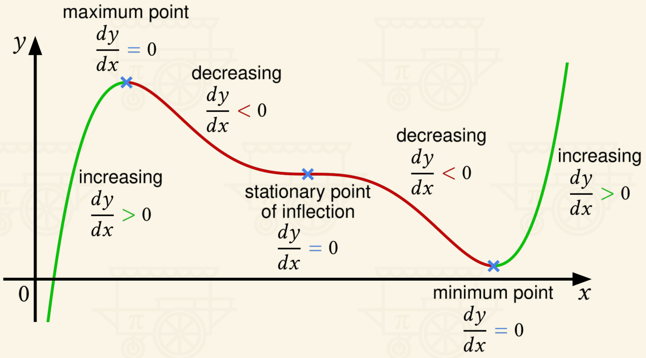
4

Graph Applications

Derivatives have major implications on graphs of functions.

Derivatives and Graphs

A function increases where its derivative is positive and decreases where it's negative. Stationary points occur where the derivative is zero, which include local minimum and maximum points, as well as stationary points of inflection.



BTW: A stationary point of inflection is a specific type of inflection point. The general case is beyond the syllabus.

Example 1:

Let

$$f(x) = x^3 + 3x^2 - 9x + 1$$

a) Find $f'(x)$.

b) Find the coordinates of the turning points of $y = f(x)$.

c) Find the regions where $f(x)$ is increasing.

d) Find the regions where $f(x)$ is decreasing.

a) – $f'(x) = 3x^2 + 6x - 9$

b) – At turning points, $f'(x) = 0$

$$\begin{aligned}\Rightarrow\quad & 3x^2 + 6x - 9 = 0\\ & x^2 + 2x - 3 = 0\\ & (x-1)(x+3) = 0\\ & x = 1 \quad\text{or}\quad x = -3\end{aligned}$$

– $f(1) = (1)^3 + 3(1)^2 - 9(1) + 1$
$= -4$

– $f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) + 1$
$= 28$

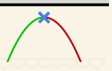
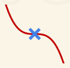
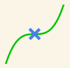
Coordinates of the turning points are $(1, -4)$ and $(-3, 28)$.

c) – $(x-1)(x+3) > 0$
$x < -3 \quad\text{or}\quad x > 1$

d) – $-3 < x < 1$

First-derivative Test

Without graphing a function, it's often difficult to determine the nature of its stationary points. This is where derivative tests come in. By examining the sign of the derivative on either side of a stationary point, we can identify whether it is a maximum point, minimum point, or stationary point of inflection.

	Just Before	At Point	Just After	Shape
Maximum Point	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$	
Minimum Point	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$	
Stationary Point of Inflection	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$	
	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$	

Example 1:

Let

$$f(x) = x^3 + 9x^2 + 24x - 16$$

Find the coordinates and nature of the stationary points of $y = f(x)$.

- $f'(x) = 3x^2 + 18x + 24$
- At the stationary points, $f'(x) = 0$.
  $3x^2 + 18x + 24 = 0$
  $(x + 2)(x + 4) = 0$
  $x = -2 \quad \text{or} \quad x = -4$
- $f'(-1) = 3(-1)^2 + 18(-1) + 24$
  $= 9$
- $f'(-3) = 3(-3)^2 + 18(-3) + 24$
  $= -3$
- $f'(-5) = 3(-5)^2 + 18(-5) + 24$
  $= 9$
- $f(-2) = (-2)^3 + 9(-2)^2 + 24(-2) - 16$
  $= -36$
- $f(-4) = (-4)^3 + 9(-4)^2 + 24(-4) - 16$
  $= -32$
- The point $(-2, -36)$ is a maximum point.
- The point $(-4, -32)$ is a minimum point.

Second-derivative Test

If the second derivative is positive, the point is a minimum; if negative, it's a maximum. This is often a quicker way to classify stationary points, but might fail to be informative.

	Derivative		
	First	Second	Shape
Maximum Point	$\dfrac{dy}{dx} = 0$	$\dfrac{d^2y}{dx^2} < 0$	
Minimum Point	$\dfrac{dy}{dx} = 0$	$\dfrac{d^2y}{dx^2} > 0$	
Uninformative	$\dfrac{dy}{dx} = 0$	$\dfrac{d^2y}{dx^2} = 0$	?

These tests are one-directional: $dy/dx = 0$ and $d^2y/dx^2 < 0$ guarantees a maximum point, but not the other way around. If $d^2y/dx^2 = 0$, the point could be a maximum, minimum, or inflection.

Example 1:

Let

$$f(x) = x^3 - 12x + 6$$

Find the coordinates and nature of the stationary points of $y = f(x)$.

- $f'(x) = 3x^2 - 12$
- At the stationary points, $f'(x) = 0$.
  $3x^2 - 12 = 0$
  $x = \pm 2$
  $x = -2 \quad \text{or} \quad x = 2$
- $f''(x) = 6x$
- $f''(-2) = 6(-2)$
  $= -12$
- $f''(2) = 6(2)$
  $= 12$
- $f(-2) = (-2)^3 - 12(-2) + 6$
  $= 22$
- $f(2) = (2)^3 - 12(2) + 6$
  $= -10$
- The point $(-2, 22)$ is a maximum point.
- The point $(2, -10)$ is a minimum point.

Lagrange's Notation

Let's convert all the identities into Lagrange's Notation.

Power Rule
- $(x^n)' = nx^{n-1}$

Scalar Multiple
- $[kf(x)]' = kf'(x)$

Addition/Subtraction Rule
- $(u + v)' = u' + v'$

Product Rule
- $(uv)' = u'v + uv'$

Quotient Rule
- $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

Chain Rule
- $[f(g(x))]' = f'(g(x))g'(x)$

Trigonometric Derivatives
- $[\sin(x)]' = \cos(x)$
- $[\cos(x)]' = -\sin(x)$
- $[\tan(x)]' = \sec^2(x)$
- $[\sin(u)]' = \cos(u)u'$
- $[\cos(u)]' = -\sin(u)u'$
- $[\tan(u)]' = \sec^2(u)u'$

Exponential Derivatives
- $(e^x)' = e^x$
- $(e^u)' = e^u u'$

Logarithmic Derivatives
- $[\ln(x)]' = \dfrac{1}{x}$
- $[\ln(u)]' = \dfrac{1}{u}u'$

Both notations are equally valid. Lagrange's notation is more common in certain areas of mathematics, while Leibniz's notation remains favored at the school level for its clarity in showing which variables are involved. Use whichever notation feels more intuitive to you.

REACH US AT:

MATHHAWKER.COM.SG – ALL NOTES

@MATHHAWKER
@MATHHAWKER – LATEST UPDATES

PATREON.COM/MATHHAWKER – SUPPORT US :)