Elementary Mathematics – Vectors

1. Vectors 2. Matrices 3. Sets

1 Vectors

Time to elevate calculus to a whole new dimension!

Introduction to Vectors

So far, we've applied calculus to single values (scalars). Vectors let us do the same with structured, multidimensional data. Vectors can either be structured vertically (called column vectors) or horizontally (called row vectors). We mainly focus on column vectors for the O-Level syllabus.

Definition

- vector $\mathbf{x}_n = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$ scalars

dimension

Example Usage

- **Fruit Vector** $= \begin{pmatrix} \text{Apples} \\ \text{Pears} \\ \text{Peaches} \\ \text{Bananas} \\ \text{Cherries} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 6 \end{pmatrix}$

Vectors are represented by a bold lowercase letter **p** on digital mediums. It's hard to draw bold symbols on paper so alternative notations are used. The standard handwritten notation is a lowercase letter with an arrow on top $\vec{p}$. Some textbooks and teachers recommend $\tilde{p}$ or $\underset{\sim}{p}$, but this is not convention. Ultimately, follow your school's convention.

Vector Operations

Vectors can be manipulated using operations that apply directly to their components.

Vector Addition

- $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$

Vector Subtraction

- $\mathbf{x} - \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \end{pmatrix}$

Scalar Multiplication

- $c\mathbf{x} = c\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix}$

Vector Equality

- $\mathbf{x} = \mathbf{y} \iff \begin{matrix} x_1 = y_1 \\ x_2 = y_2 \\ x_3 = y_3 \end{matrix}$

Example 1:

Evaluate the expression $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

– $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$

Example 2:

Li Yun is in charge of managing the inventory at her school's co-op. She decides to represent the stock using a vector, where the entries correspond to the number of pens, notebooks, and rulers, in that order.

1. The co-op receives an order of 100 pens, 50 notebooks, and 30 rulers.
2. After a large sale, 82 pens, 24 notebooks, and 18 rulers are sold.
3. A fire damages the store, destroying half of the remaining stock.
4. To prepare for the new school year, a second order double the size of the initial order is placed.

– $\begin{pmatrix} \text{pens} \\ \text{notebooks} \\ \text{rulers} \end{pmatrix} = \left[\begin{pmatrix} 100 \\ 50 \\ 30 \end{pmatrix} - \begin{pmatrix} 82 \\ 24 \\ 18 \end{pmatrix}\right] \times \frac{1}{2} + 2\begin{pmatrix} 100 \\ 50 \\ 30 \end{pmatrix} = \begin{pmatrix} 209 \\ 113 \\ 66 \end{pmatrix}$

Example 3:

Find the value of k so that $\begin{pmatrix} 3 \\ -9 \end{pmatrix} = k\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

– $\begin{pmatrix} 3 \\ -9 \end{pmatrix} = k\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$-3\begin{pmatrix} -1 \\ 3 \end{pmatrix} = k\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$k = -3$

Example 4:

Solve the equation $x\begin{pmatrix} 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

– $x\begin{pmatrix} 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 2x + y - 7 \\ 3x - y - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2x + y - 7 = 0$ — (1)

$3x - y - 3 = 0$ — (2)

– (1) + (2):

$5x - 10 = 0$

$x = 2$

– Sub $x = 2$ into (1):

$2(2) + y - 7 = 0$

$y = 3$

Example 5:

Solve the equation $\begin{pmatrix} 3y - 2 \\ x + 1 \end{pmatrix} + \begin{pmatrix} 2x - 4 \\ 5 - y \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$.

– $\begin{pmatrix} 3y - 2 \\ x + 1 \end{pmatrix} + \begin{pmatrix} 2x - 4 \\ 5 - y \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$

$\begin{pmatrix} 3y + 2x - 17 \\ x - y - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3y + 2x - 17 = 0$ — (1)

$x - y - 1 = 0$ — (2)

– (1) + 3(2):

$5x - 20 = 0$

$x = 4$

– Sub $x = 4$ into (2):

$4 - y - 1 = 0$

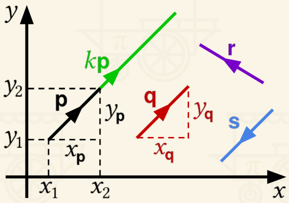
$y = 3$

Coordinate Vectors

When we place vectors in a coordinate system, they gain direction, length, and angle. This turns the abstract concept of vectors into something we can use to describe real-world ideas like movement, force, and position.

Definition

- $\mathbf{p} = \begin{pmatrix} x_\mathbf{p} \\ y_\mathbf{p} \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$



Magnitude

- Magnitude of vector $\mathbf{p}$ is its length and represented by $|\mathbf{p}|$.
- $|\mathbf{p}| = \sqrt{x_\mathbf{p}^2 + y_\mathbf{p}^2}$
- $|\mathbf{p} + \mathbf{r}| \neq |\mathbf{p}| + |\mathbf{r}|$
- $|k\mathbf{p}| = |k||\mathbf{p}| = |k|\sqrt{x_\mathbf{p}^2 + y_\mathbf{p}^2}$, where k is some scalar

Negative Vectors

- If $\mathbf{s}$ has the same magnitude but opposite directions as $\mathbf{p}$, $\mathbf{s}$ is the negative vector of $\mathbf{p}$.
- $\mathbf{s} = -\mathbf{p}$
- $\mathbf{p} + \mathbf{s} = \mathbf{0}$

Parallel Vectors

- Vectors are parallel if they have the same or opposite directions.
- $\mathbf{p} \parallel \mathbf{q} \parallel \mathbf{s} \parallel k\mathbf{p}$, where k is some non-zero scalar

Vectors are fully defined by their magnitude and direction, not by their position in space. This makes them "free-floating". For example, vectors $\mathbf{p}$ and $\mathbf{q}$ are equivalent since they have the same length and they point in the same direction.

Example 1:

It is given that vectors $\mathbf{p} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 15 \\ 20 \end{pmatrix}$.

a) Find the vectors (if any) that are equal in magnitude.

b) Find the vectors (if any) that are equal in direction, and justify it using scalar multiples.

a) – $|\mathbf{p}| = \sqrt{7^2 + 24^2} = 25$

$|\mathbf{r}| = \sqrt{15^2 + 20^2} = 25$

– $|\mathbf{p}| = |\mathbf{r}|$

$\Rightarrow$ Vectors $\mathbf{p}$ and $\mathbf{r}$ have the same magnitude.

b) – $\mathbf{q} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} = -1 \times \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5 \times \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

– $\frac{1}{-1}\mathbf{q} = \frac{1}{5}\mathbf{r}$

$\Rightarrow$ Vectors $\mathbf{q}$ and $\mathbf{r}$ have the same direction.

Example 2:

Vector $\mathbf{q}$ is parallel to vector $\mathbf{p} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, point in the same direction, and has a magnitude of 10.

a) Find the magnitude of $\mathbf{p}$.

b) Determine the scalar k such that $\mathbf{p} = k\mathbf{q}$.

c) Find vector $\mathbf{q}$.

a) – $|\mathbf{p}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

b) – $\mathbf{q} \parallel \mathbf{p}$

$\Rightarrow$ $\mathbf{q} = k\mathbf{p}$

– $|\mathbf{q}| = |k\mathbf{p}| = |k||\mathbf{p}|$

$10 = |k|\sqrt{5}$

$|k| = 2\sqrt{5}$

$k = 2\sqrt{5}$ or $k = -2\sqrt{5}$

– $\mathbf{q}$ points in the same direction as $\mathbf{p}$.

$\Rightarrow$ $k = 2\sqrt{5}$

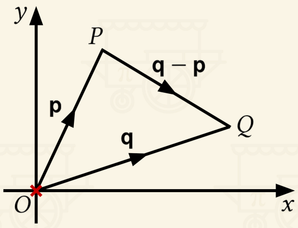
c) – $\mathbf{q} = k\mathbf{p} = 2\sqrt{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4\sqrt{5} \\ -2\sqrt{5} \end{pmatrix}$

Position and Displacement Vectors

While vectors are not inherently tied to any location, they can be used to describe location-specific phenomena by anchoring them to a fixed point like the origin.

Position Vectors

- $\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix} x_\mathbf{p} \\ y_\mathbf{p} \end{pmatrix}$
- $\overrightarrow{OQ} = \mathbf{q} = \begin{pmatrix} x_\mathbf{q} \\ y_\mathbf{q} \end{pmatrix}$
- $\overrightarrow{PO} = -\overrightarrow{OP} = -\mathbf{p}$



Displacement Vectors

- Vectors can be added sequentially by aligning them tip-to-tail, where the terminal point of one vector matches the initial point of the next.
- $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = -\begin{pmatrix} x_\mathbf{p} \\ y_\mathbf{p} \end{pmatrix} + \begin{pmatrix} x_\mathbf{q} \\ y_\mathbf{q} \end{pmatrix} = \begin{pmatrix} x_\mathbf{q} - x_\mathbf{p} \\ y_\mathbf{q} - y_\mathbf{p} \end{pmatrix}$

Position vectors are always referenced from the origin. Displacement vectors tell you how to go from one point to another, regardless of the origin. Displacement Vectors are also sometimes called Translation Vectors if used in the context of geometry.

Example 1:

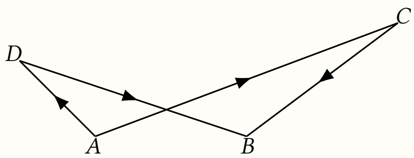
Given points $P(-1, 2)$ and $Q(6, 3)$, write down position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Hence, find $\overrightarrow{PQ}$.

– $\overrightarrow{OP} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ – $\overrightarrow{OQ} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

– $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = -\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

Example 2:

Given $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and $\overrightarrow{DB} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$, find $\overrightarrow{AD}$.



– $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

$= \overrightarrow{AC} + \overrightarrow{CB} - \overrightarrow{DB}$

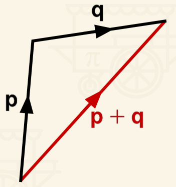
$= \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Addition and Subtraction Revisited

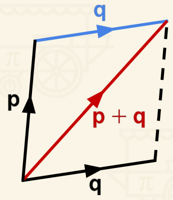
Adding and subtracting vectors is easy. Figuring out the arrow of the resultant vector is easy too with a few simple tricks!

Case 1: Triangle Law of Vector Addition



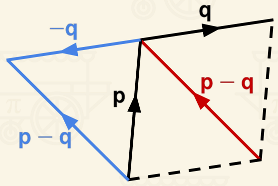
Sum of vectors aligned tip-to-tail form a displacement vector from the start to the end.

Case 2: Parallelogram Law of Vector Addition



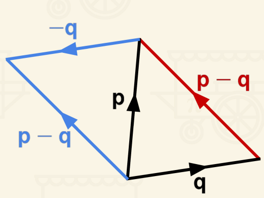
Equivalent to Case 1 considering "free-floating" nature of vectors.

Case 3: Parallelogram Law of Vector Subtraction*



Extension of Case 2 with a negative vector

Case 4: Triangle Law of Vector Subtraction

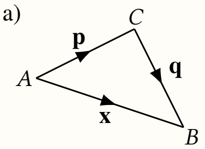


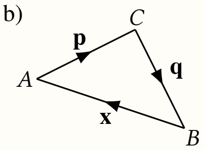
Extension of Case 1 with negative a vector.

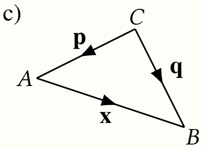
*unlike the other three, the Parallelogram Law of Vector Subtraction isn't formally recognized as a standard law. Ultimately, these laws are just visual tools to aid understanding. Once you grasp how to translate vectors, reverse their direction when needed, and chain them together, determining the resultant displacement vector becomes straightforward.

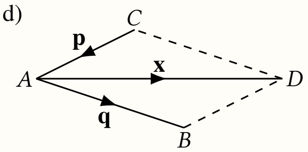
Example 1:

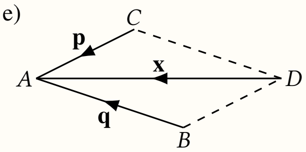
For each of the following figures, find $\mathbf{x}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

a)


b)


c)


d)


e)


a) – $\mathbf{x} = \mathbf{p} + \mathbf{q}$

b) – $\mathbf{x} = -\mathbf{p} - \mathbf{q}$

c) – $\mathbf{x} = \mathbf{q} - \mathbf{p}$

d) – $\mathbf{x} = \mathbf{q} - \mathbf{p}$

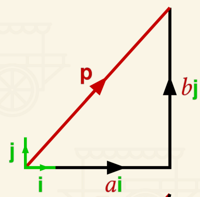
e) – $\mathbf{x} = \mathbf{p} + \mathbf{q}$

Vector Decomposition

Any vector can be broken into horizontal and vertical components that form a right-angled triangle. More generally, any 2-dimensional vector can be expressed using any pair of non-parallel vectors.

Base Case

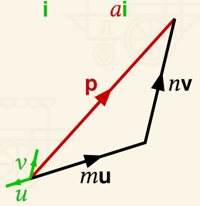
- The simplest example is constructing vectors using base vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\mathbf{i} + b\mathbf{j}$



General Case

- $\mathbf{p} = m\mathbf{u} + n\mathbf{v}$

  where m, n are real numbers and u, v are non-zero and non-parallel vectors.



Example 1:

Vector $\mathbf{p} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ can be written as a combination of $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Write $\mathbf{p} = a\mathbf{u} + b\mathbf{v}$ where a and b are real numbers to be found.

– $\begin{pmatrix} 6 \\ 4 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 2:

A drone starts at the origin and follows these moves:

1. 3 units in the direction of $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
2. 2 units in the direction of $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

a) Calculate the drone's total movement $\mathbf{p}$.

b) Is $\mathbf{p}$ a position vector or a displacement vector?

c) Could $\mathbf{p}$ be composed from $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$?

a) – $\mathbf{p} = 3\mathbf{u} + 2\mathbf{v} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

b) – Position vector.

c) – No. There is no a and b such that $\mathbf{p} = a\mathbf{x} + b\mathbf{y}$.

Example 3:

In $\triangle ABC$, points D and E are the midpoints of AC and BC, respectively. Let $\mathbf{p} = \overrightarrow{CD}$ and $\mathbf{q} = \overrightarrow{CE}$.

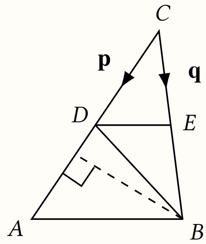
a) Show that $\overrightarrow{DE} = \mathbf{q} - \mathbf{p}$.

b) Show that $\overrightarrow{DE} \parallel \overrightarrow{AB}$.

c) Prove that $\triangle ABC$ is similar to $\triangle DEC$.

d) Find $\dfrac{\text{Area}_{\triangle DEC}}{\text{Area}_{\triangle ABC}}$.

e) Find $\dfrac{\text{Area}_{\triangle ABD}}{\text{Area}_{\triangle ABC}}$.



a) – $\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE} = \mathbf{q} - \mathbf{p}$

b) – $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

$= -2\mathbf{p} + 2\mathbf{q}$

$= 2(\mathbf{q} - \mathbf{p})$

$= 2\overrightarrow{DE}$

$\Rightarrow \overrightarrow{DE} \parallel \overrightarrow{AB}$

c) – $\dfrac{\overrightarrow{CD}}{\overrightarrow{CA}} = \dfrac{\overrightarrow{CE}}{\overrightarrow{CB}} = \dfrac{1}{2}$

– $\angle ACB = \angle DCE$ (common angle)

$\therefore$ $\triangle ABC \sim \triangle DEC$ (SAS similarity test)

d) – $\dfrac{\text{Area}_{\triangle DEC}}{\text{Area}_{\triangle ABC}} = \left(\dfrac{DE}{AB}\right)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

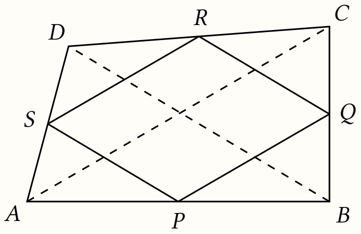
e) – $\dfrac{\text{Area}_{\triangle ABD}}{\text{Area}_{\triangle ABC}} = \dfrac{\frac{1}{2} \times \overrightarrow{DA} \times h}{\frac{1}{2} \times \overrightarrow{CA} \times h} = \dfrac{\overrightarrow{DA}}{\overrightarrow{CA}} = \dfrac{\mathbf{p}}{2\mathbf{p}} = \dfrac{1}{2}$

Example 4:

In the quadrilateral $ABCD$, points P, Q, R, and S are the midpoints of AB, BC, CD, and DA, respectively.

a) Show that $BD = 2PS$.

b) Show that $PQRS$ is a parallelogram.



a) – Let $\overrightarrow{AP} = \mathbf{a}$
$\overrightarrow{AS} = \mathbf{b}$

– $\overrightarrow{PS} = \overrightarrow{PA} + \overrightarrow{AS} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

– $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -2\mathbf{a} + 2\mathbf{b} = 2(\mathbf{b} - \mathbf{a}) = 2\overrightarrow{PS}$

– $|\overrightarrow{BD}| = |2\overrightarrow{PS}|$
$|\overrightarrow{BD}| = |2||\overrightarrow{PS}|$
$BD = 2PS$

b) – Let $\overrightarrow{CR} = \mathbf{c}$
$\overrightarrow{CQ} = \mathbf{d}$

– $\overrightarrow{QR} = \overrightarrow{QC} + \overrightarrow{CR} = -\mathbf{d} + \mathbf{c} = \mathbf{c} - \mathbf{d}$

– $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -2\mathbf{d} + 2\mathbf{c} = 2(\mathbf{c} - \mathbf{d}) = 2\overrightarrow{QR}$

– $\overrightarrow{BD} = 2\overrightarrow{PS} = 2\overrightarrow{QR}$
$\overrightarrow{PS} = \overrightarrow{QR}$

$\Rightarrow$ PS and QR are parallel and have the same length.

$\Rightarrow$ $PQRS$ is a parallelogram.

2 Matrices

Why stop now? Let's add yet another dimension!

Introduction to Matrices

Matrices are an extension of vectors, arranged in rows and columns instead of a single column. Most of the operations work the same as for vectors.

Definition

$$\text{matrix} \quad \mathbf{X}_{3\times 3} = \begin{array}{c} \text{columns} \\ \begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} \text{rows}$$

order

scalars

Matrix Addition

- $\mathbf{X} + \mathbf{Y} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + y_{11} & x_{12} + y_{12} \\ x_{21} + y_{21} & y_{22} + y_{22} \end{pmatrix}$

Matrix Subtraction

- $\mathbf{X} - \mathbf{Y} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11} - y_{11} & x_{12} - y_{12} \\ x_{21} - y_{21} & y_{22} - y_{22} \end{pmatrix}$

Scalar Multiplication

- $c\mathbf{X} = c\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} cx_{11} & cx_{12} \\ cx_{21} & cx_{22} \end{pmatrix}$

Matrix Equality

- $\mathbf{x} = \mathbf{y} \quad \Leftrightarrow \quad \begin{array}{ll} x_{11} = y_{11} & x_{12} = y_{12} \\ x_{21} = y_{21} & x_{22} = y_{22} \end{array}$

Matrices are represented by a bold uppercase letter **M** when printed and italic uppercase letter M when written.

Example 1:

The table below shows Li Yun's record of sales at the school co-op over one week.

Item	Mon	Tue	Wed	Thu	Fri
Pens	120	135	140	130	125
Notebooks	80	75	90	85	95
Rulers	50	60	55	52	58

a) Represent the data using a matrix **M**.

b) State the order of **M**.

c) Find the total number of pens sold that week.

d) Find the total number items sold on Tuesday.

a) – $\mathbf{M} = \begin{pmatrix} 120 & 135 & 140 & 130 & 125 \\ 80 & 75 & 90 & 85 & 95 \\ 50 & 60 & 55 & 52 & 58 \end{pmatrix}$

b) – The order **M** is 3×5.

c) – Pens Sold $= 120 + 135 + 140 + 130 + 125 = 650$

d) – Items Sold $= 135 + 75 + 60 = 270$

Example 2:

Given matrices $\mathbf{M} = \begin{pmatrix} 2a & -2 \\ 5+c & -b \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 14 & b \\ a-3c & a-5 \end{pmatrix}$, find the values of a, b, and c such that $\mathbf{M} = \mathbf{N}$.

– $\begin{pmatrix} 2a & -2 \\ 5+c & -b \end{pmatrix} = \begin{pmatrix} 14 & b \\ a-3c & a-5 \end{pmatrix}$

– $2a = 14$
$a = 7$

– $b = -2$

– $5 + c = a - 3c$
$4c = 7 - 5$
$c = \frac{1}{2}$

Special Matrices

Just like how numbers like 1 and 0 have special roles in arithmetic, the identity and zero matrices play important roles in matrix operations.

Identity Matrix

- A square matrix with ones on the diagonal and zeroes everywhere else.

- $\mathbf{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$

Zero Matrix

- A square matrix with zeroes everywhere.

- $\mathbf{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$

Example 1:

The tables below shows Li Yun's record of sales at the school co-op over two weeks.

Week 1

Item	Mon	Tue	Wed	Thu	Fri
Pens	120	135	140	130	125
Notebooks	80	75	90	85	95
Rulers	50	60	55	52	58

Week 2

Item	Mon	Tue	Wed	Thu	Fri
Pens	110	140	135	120	130
Notebooks	89	71	91	78	88
Rulers	48	62	50	55	60

a) Represent the data using **M** and **N** for weeks 1 and 2, respectively.

b) Find $\mathbf{X} = \mathbf{M} + \mathbf{N}$.

c) Find $\mathbf{Y} = \mathbf{N} - \mathbf{M}$.

d) Describe what matrices **X** and **Y** represent.

a) – $\mathbf{M} = \begin{pmatrix} 120 & 135 & 140 & 130 & 125 \\ 80 & 75 & 90 & 85 & 95 \\ 50 & 60 & 55 & 52 & 58 \end{pmatrix}$

– $\mathbf{N} = \begin{pmatrix} 110 & 140 & 135 & 120 & 130 \\ 89 & 71 & 91 & 78 & 88 \\ 48 & 62 & 50 & 55 & 60 \end{pmatrix}$

b) – $\mathbf{X} = \mathbf{M} + \mathbf{N}$

$$= \begin{pmatrix} 120 & 135 & 140 & 130 & 125 \\ 80 & 75 & 90 & 85 & 95 \\ 50 & 60 & 55 & 52 & 58 \end{pmatrix} + \begin{pmatrix} 110 & 140 & 135 & 120 & 130 \\ 89 & 71 & 91 & 78 & 88 \\ 48 & 62 & 50 & 55 & 60 \end{pmatrix}$$

$$= \begin{pmatrix} 230 & 275 & 275 & 250 & 255 \\ 169 & 146 & 181 & 163 & 183 \\ 98 & 122 & 105 & 107 & 118 \end{pmatrix}$$

c) – $\mathbf{Y} = \mathbf{N} - \mathbf{M}$

$$= \begin{pmatrix} 110 & 140 & 135 & 120 & 130 \\ 89 & 71 & 91 & 78 & 88 \\ 48 & 62 & 50 & 55 & 60 \end{pmatrix} - \begin{pmatrix} 120 & 135 & 140 & 130 & 125 \\ 80 & 75 & 90 & 85 & 95 \\ 50 & 60 & 55 & 52 & 58 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 5 & -5 & -10 & 5 \\ 9 & -4 & 1 & 5 & -7 \\ -2 & 2 & -5 & 3 & 2 \end{pmatrix}$$

d) – $\mathbf{X}$ represents the total number of each item sold each day across the two weeks. $\mathbf{Y}$ represents the increase in sales from week 1 to week 2.

Matrix Multiplication

Matrix multiplication in particular is not as simple as the other operations. There are a few steps to follow.

Step 1: Order Check

- Columns of the first matrix must match rows of the second. The result takes rows from the first matrix and columns from the second.
- $\mathbf{A}_{m\times p} \times \mathbf{B}_{p\times n} = \mathbf{C}_{m\times n}$

Step 2: Multiply and Add

- $$\mathbf{A}_{2\times 3} \times \mathbf{B}_{3\times 2} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \times \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix} = \begin{pmatrix} ag + bi + ck & ah + bj + cl \\ dg + ei + fk & dh + ej + fl \end{pmatrix}$$

Property: Non-commutative

- In general, $A \times B \neq B \times A$.

Example 1:

It is given that matrices $\mathbf{A} = \begin{pmatrix} 7 & -3 \\ 2 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 6 \\ -1 & 8 \end{pmatrix}$.

a) Evaluate $\mathbf{AB}$.

b) Evaluate $\mathbf{BA}$.

a) – $\mathbf{AB} = \begin{pmatrix} 7 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ -1 & 8 \end{pmatrix}$

$$= \begin{pmatrix} 7\times 4 + -3\times -1 & 7\times 6 + -3\times 8 \\ 2\times 4 + 5\times -1 & 2\times 6 + 5\times 8 \end{pmatrix}$$

$$= \begin{pmatrix} 31 & 18 \\ 3 & 52 \end{pmatrix}$$

b) – $\mathbf{BA} = \begin{pmatrix} 4 & 6 \\ -1 & 8 \end{pmatrix}\begin{pmatrix} 7 & -3 \\ 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 4\times 7 + 6\times 2 & 4\times -3 + 6\times 5 \\ -1\times 7 + 8\times 2 & -1\times -3 + 8\times 5 \end{pmatrix}$$

$$= \begin{pmatrix} 40 & 18 \\ 9 & 43 \end{pmatrix}$$

Example 2:

The tables below shows Li Yun's record of sales at the school co-op over one week as well as the respective prices of the items.

Item	Mon	Tue	Wed	Thu	Fri
Pens	120	135	140	130	125
Notebooks	80	75	90	85	95
Rulers	50	60	55	52	58

	Pen	Notebook	Ruler
Prices ($)	1.50	2.50	0.80

a) Represent the data using matrices $\mathbf{M}$ and $\mathbf{P}$ of orders 3×5 and 1×3, respectively.

b) Find $\mathbf{R} = \mathbf{PM}$.

c) Describe what matrix $\mathbf{R}$ represents. Hence, determine which day had the best sales.

d) Find a matrix $\mathbf{O}$ such that $r = \mathbf{RO}$, where r is a scalar that represents the total revenue of the week. Hence, find r.

a) – $\mathbf{M} = \begin{pmatrix} 120 & 135 & 140 & 130 & 125 \\ 80 & 75 & 90 & 85 & 95 \\ 50 & 60 & 55 & 52 & 58 \end{pmatrix}$

– $\mathbf{P} = \begin{pmatrix} 1.50 & 2.50 & 0.80 \end{pmatrix}$

b) – $\mathbf{R} = \mathbf{PM}$

$$= \begin{pmatrix} 1.50 & 2.50 & 0.80 \end{pmatrix}\begin{pmatrix} 120 & 135 & 140 & 130 & 125 \\ 80 & 75 & 90 & 85 & 95 \\ 50 & 60 & 55 & 52 & 58 \end{pmatrix}$$

$$= \begin{pmatrix} 420.0 & 438.0 & 479.0 & 449.1 & 471.4 \end{pmatrix}$$

c) – $\mathbf{R}$ represents the daily revenue of the co-op.

– Wednesday had the best sales at \$479.00.

d) – Let $\mathbf{O} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

– $r = \mathbf{RO}$

$$= \begin{pmatrix} 420.0 & 438.0 & 479.0 & 449.1 & 471.4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \$2257.50$$

3 Sets

Vectors and matrices structure data. Sets just collect it.

Introduction to Sets

A set is simply a collection of distinct objects. These objects are called the elements or members of the set. There are a few ways to represent sets.

Roster Notation

- List out all elements:

$\{1, 3, 5, 7, 9\}$

Set Builder Notation

- Describe rule for elements:

$\{\underbrace{x}_{\text{element description}} \mid \underbrace{x \text{ is an odd number} \leq 10}_{\text{conditions}}\}$

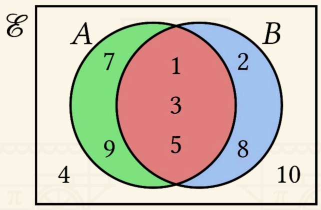
Venn Diagrams

- Venn diagrams represent sets as circles to visualize relationships between them.

$A = \{x \mid x \text{ is an odd number} \leq 10\}$
$B = \{x \mid x \text{ is a Fibonacci number} \leq 10\}$

- The universal set $\mathscr{E}$ contains all elements under consideration. It is represented by the rectangle that surrounds all the circles.

$\mathscr{E} = \{x \mid x \text{ is an integer between 1 and 10, inclusively}\}$



You might notice that there are many ways to write the rules in the set builder notation with different levels of precision. Generally, more detailed the better, but here we'll focus on clarity and simplicity. Also, the standard notation for the universal set is actually U, but the Cambridge syllabus uses the symbol $\mathscr{E}$. We'll, therefore, stick to $\mathscr{E}$ (read "Script E" or just "E") in these notes.

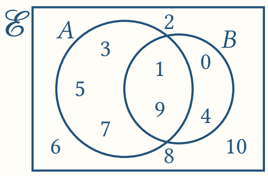
Example 1:

It is given $\mathscr{E} = \{x \mid x \text{ is an integer between 0 and 10 inclusively}\}$, $A = \{x \mid x \text{ is odd}\}$, and $B = \{x \mid x \text{ is a perfect square}\}$, where $\mathscr{E}$ contains A and B.

a) Express $\mathscr{E}$, A, and B in roster notation.
b) Draw a Venn diagram to represent $\mathscr{E}$, A, and B.

a) – $\mathscr{E} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{1, 3, 5, 7, 9\}$
$B = \{0, 1, 4, 9\}$

b)



Additional Notation

There are several additional notations that work alongside set notation to help express ideas more clearly and precisely in mathematical language.

1. $\mathbb{N}$: represents the set of all natural numbers.
2. $\mathbb{Z}$: represents the set of all integers numbers.
3. $\mathbb{R}$: represents the set of all real numbers.
4. $\varnothing$: is called the empty set that contains no elements.
5. $x \in A$: means "x is an element of set A".
6. $x \notin A$: means "x is not an element of set A".
7. $n(A)$: represents the number of elements in set A.
8. $A = B$: means "sets A and B contain the same elements".
9. $A \neq B$: means "sets A and B do not contain the same elements".
10. $A \subseteq B$: subset, means "all elements of set A are in B".
11. $A \subset B$: proper subset, means "all elements of set A are in set B, but $A \neq B$".

Example 1:

It is given that $A = \{x \in \mathbb{Z} \mid x = 2k + 1, 0 \leq k \leq 2\}$,
$B = \{x \in \mathbb{Z} \mid x \text{ is in the Fibonacci sequence and } 1 < x \leq 5\}$,
$C = \{x \in \mathbb{Z} \mid x \text{ is a prime number and } x \leq 10\}$.

a) Express A, B, and C in roster notation.
b) State $n(A)$, $n(B)$, and $n(C)$.
c) State any proper subsets that exist among A, B, and C.

a) – $A = \{1, 3, 5\}$
$B = \{2, 3, 5\}$
$C = \{2, 3, 5, 7\}$

b) – $n(A) = 3$
$n(B) = 3$
$n(C) = 4$

c) – $B \subset C$

Example 2:

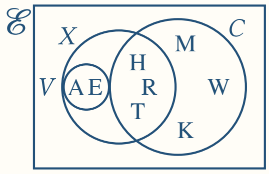
It is given that $\mathscr{E}$ is the set of letters used to form the word 'MATH HAWKER', V is the set of vowels in $\mathscr{E}$, C is the set of consonants in $\mathscr{E}$, and X is the set of letters used to form the word 'HEART'.

a) Express $\mathscr{E}$, V, C, and X in roster notation.
b) State $n(\mathscr{E})$, $n(V)$, $n(C)$, and $n(X)$.
c) Draw a Venn diagram to represent $\mathscr{E}$, V, C, and X.

a) – $\mathscr{E} = \{\mathrm{M, A, T, H, W, K, E, R}\}$
$V = \{\mathrm{A, E}\}$
$C = \{\mathrm{M, T, H, W, K, R}\}$
$X = \{\mathrm{H, E, A, R, T}\}$

b) – $n(\mathscr{E}) = 8$
$n(V) = 2$
$n(C) = 6$
$n(X) = 5$

c)

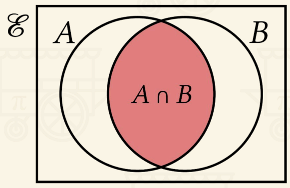


Set Operations

In set theory, we often need to combine, compare, or separate sets to understand how they relate to one another. This is where set operations come in.

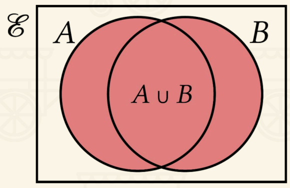
Intersection

- Finds elements common to both sets.



Union

- Combines all elements from two sets.

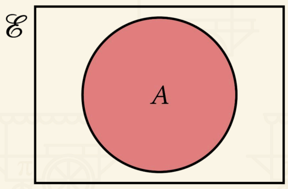


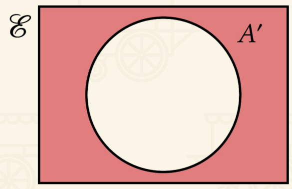
Inclusion-Exclusion Principle

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Complement

The complement of a set contains all elements in the universal set that are not in said set. The complement of set A is A' read as "A prime".





Example 1:

It is given that $\mathscr{E} = \{x \in \mathbb{Z} \mid 1 \leq x \leq 15\}$,
$A = \{x \in \mathscr{E} \mid x \text{ is even}\}$,
$B = \{x \in \mathscr{E} \mid x \text{ is a factor of } 12\}$,
$C = \{x \in \mathscr{E} \mid x \text{ is a prime number}\}$.

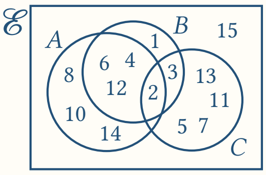
a) Express $\mathscr{E}$, A, B, and C in roster notation.
b) Draw a Venn diagram to represent $\mathscr{E}$, A, B, and C.

c) Express the following expressions in roster notation and a shaded Venn diagram.

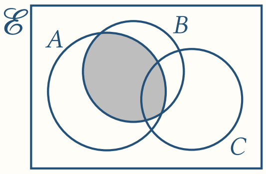
i. $A \cap B$ ii. $(A \cup B \cup C)'$ iii. $(A \cup B) \cap C'$
iv. $(A \cap C) \cap B'$ v. $[(A \cap B) \cup (B \cap C)] \cap (A \cap B \cap C)'$

a) – $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$
$A = \{2, 4, 6, 8, 10, 12, 14\}$
$B = \{1, 2, 3, 4, 6, 12\}$
$C = \{2, 3, 5, 7, 11, 13\}$

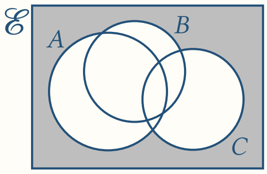
b)



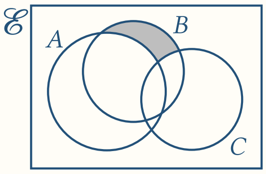
c) i. – $A \cap B = \{2, 4, 6, 12\}$



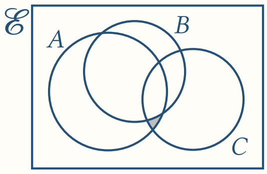
ii. – $(A \cup B \cup C)' = \{15\}$



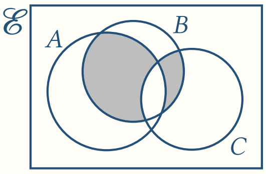
iii. – $(A \cup C)' \cap B = \{1\}$



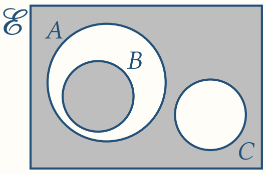
iv. – $(A \cap C) \cap B' = \varnothing$



v. – $[(A \cap B) \cup (B \cap C)] \cap (A \cap B \cap C)' = \{3, 4, 6, 12\}$



Example 2:
Describe the shaded region of the Venn diagram in set notation.



– $(A \cup C)' \cup B$

Example 3:
It is given that $A = \{(x, y) \mid y = x^2 - 4x + 3, x, y \in \mathbb{R}\}$
$B = \{(x, y) \mid y = 0, x, y \in \mathbb{R}\}$
Express $A \cap B$ in roster notation.

– $y = x^2 - 4x + 3$ — (1)
$y = 0$ — (2)
– Sub (1) into (2):
$x^2 - 4x + 3 = 0$
$(x - 3)(x - 1) = 0$
$x = 3$ or $x = 1$
– $A \cap B = \{(3, 0), (1, 0)\}$

Example 4:
It is given that $A = \{(x, y) \mid (x, y)$ is a point on the curve $y = x^2 - 2x + 4\}$
$B = \{(x, y) \mid (x, y)$ is a point on the curve $y = x - 1\}$
Express $A \cap B$ in roster notation.

– $y = x^2 - 2x + 4$ — (1)
$y = 3x - 1$ — (2)
– Sub (1) into (2):
$x^2 - 2x + 4 = x - 1$
$x^2 - 3x + 5 = 0$
– $(-3)^2 - 4(1)(5) = -11 < 0$
$\Rightarrow$ There are no real roots.
– $A \cap B = \varnothing$

Example 5:
It is given $n(\mathscr{E}) = 30$, $n(A) = 12$, and $n(B) = 20$.
a) Find the maximum value of $n(A \cap B)$.
b) Find the minimum value of $n(A \cap B)$.
c) Find the maximum value of $n(A \cup B)$.
d) Find the maximum value of $n(A \cup B)'$.
e) Find the maximum value of $n(A \cap B)'$.
f) Find the maximum number of elements only in A.

a) – $n(A \cap B)$ is largest when the larger set contains the smaller.
$\Rightarrow$ $A \subseteq B$
$\Rightarrow$ $A \cap B = A$
$n(A \cap B) = n(A) = 12$

b) – $n(A \cap B)$ is smallest when $A \cup B = \mathscr{E}$.
$n(A \cap B) = n(A) + n(B) - n(\mathscr{E})$
$= 12 + 20 - 30$
$= 2$

c) – $n(A \cup B)$ is largest when $A \cup B = \mathscr{E}$.
$\Rightarrow$ $n(A \cup B) = n(\mathscr{E}) = 30$

d) – $n(A \cup B)'$ is largest when $n(A \cup B)$ is smallest.
$n(A \cup B)$ is smallest when the larger set contains the smaller.
$\Rightarrow$ $A \subseteq B$
$\Rightarrow$ $A \cup B = B$
$n(A \cup B)' = n(B)'$
$= n(\mathscr{E}) - n(B)$
$= 30 - 20$
$= 10$

e) – $n(A \cap B)'$ is largest when $n(A \cap B)$ is smallest.
$n(A \cap B)$ is smallest when $A \cup B = \mathscr{E}$.
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (inclusion-exclusion principle)
$n(\mathscr{E}) = n(A) + n(B) - n(A \cap B)$
$n(A \cap B) = n(A) + n(B) - n(\mathscr{E})$
$= 12 + 20 - 30$
$= 2$
– $n(A \cap B)' = n(\mathscr{E}) - n(A \cap B)$
$= 30 - 2$
$= 28$

f) – Number of elements only in A can be represented by $n(A \cap B')$.
$n(A \cap B')$ is largest when $n(A \cap B)$ is smallest.
$n(A \cap B)$ is smallest when $A \cup B = \mathscr{E}$.
$n(A \cap B) = n(A) + n(B) - n(\mathscr{E})$
$= 12 + 20 - 30$
$= 2$
– $n(A \cap B') = n(A) - n(A \cap B)$
$= 12 - 2$
$= 10$

Example 7:

A survey was conducted among 50 office employees about their lunch habits. 20 employees usually eat at the company canteen, 35 employees usually bring their own lunch, and 10 employees do neither. How many employees use only one of the two lunch options but not both?

- Let $\mathscr{E}$ represent all the employees in the office,
  C the employees that eat at the canteen, and
  O the employees that bring their own lunch.
- $n(C \cup O) = n(\mathscr{E}) - n(C \cup O)'$
  $= 50 - 10$
  $= 40$
- $n(C \cup O) = n(C) + n(O) - n(C \cap O)$
  $n(C \cap O) = n(C) + n(O) - n(C \cup O)$
  $= 20 + 35 - 40$
  $= 15$
- The number of employees that use only one of the two lunch options but not both can be represented by $n(C \cup O) - n(C \cap O)$.
  $n(C \cup O) - n(C \cap O) = 40 - 15 = 25$

Example 8:

In a class of 40 students, 34 students take mathematics and 28 students take science.

a) Find the smallest possible number of students who take both mathematics and science.

b) Find the largest possible number of students who take neither mathematics nor science.

a)
- Let $\mathscr{E}$ represent all the students in the class,
  M the students taking mathematics, and
  S the students taking science.
- $n(M \cap S)$ is smallest when $M \cup S = \mathscr{E}$.
  $n(M \cup S) = n(M) + n(S) - n(M \cap S)$
  $n(M \cap S) = n(M) + n(S) - n(\mathscr{E})$
  $= 34 + 28 - 40$
  $= 22$

b)
- $n(M \cup S)'$ is largest when $n(M \cup S)$ is smallest.
  $n(M \cup S)$ is smallest when the larger set contains the smaller.

$\Rightarrow$ $S \subseteq M$

$\Rightarrow$ $S \cup M = M$

$n(M \cup S)' = n(M)'$
$= n(\mathscr{E}) - n(M)$
$= 40 - 34$
$= 6$